Math 2534 Solutions to worksheet on Chap 8

1) Let $A = \mathbb{Z}^+ \times \mathbb{Z}^+$. Define a relation $R$ on $A$ as follows: for all $(a, b)$ and $(c, d)$ in $A$, $(a, b) R (c, d)$ if (Correction) $a + d = b + c$. Prove that $R$ is an equivalence relation on $A$.

Proof: Reflexive: $\forall (a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, $(a, b) R (a,b)$.
you need to verify that $a + b = b + a$.
Symmetric: For $(a,b)$ and $(c, d)$ If $(a, b) R (c, d)$ then $(c, d) R (a, b)$.
You can assume that since $(a, b) R (c, d)$ we have that $a + d = b + c$. You need to verify that $c + b = d + a$.
Transitive: For $(a, b)$, $(c, d)$ and $(e, f)$, If $(a, b) R (c, d)$ and $(c, d) R (e, f)$ then $(a, b) R (e, f)$. We will assume $(a, b) R (c, d)$ which means that $a + d = b + c$, and we will assume that $(c, d) R (e, f)$ which means that $c + f = d + e$. We need to show that $a + f = b + e$.

2) The relation $R$ defined on all sets is an equivalence relation when ARB if and only if there is a bijection $f$ so that $f : A \rightarrow B$.

Proof: For $R$ to be an equivalence relation we must verify that $R$ is reflexive, symmetric and transitive.
To verify that $R$ is reflexive we must show that for each set $A$, $A R A$. Consider the identity mapping $f(a_i) = a_i$ for each $i$. This is clearly one to one and onto and therefore is a bijection from $A$ to itself. Therefore $R$ is reflexive.

To verify that $R$ is symmetric we must show that for any arbitrary sets $A$ and $B$, IF ARB, THEN BRA. We assume that ARB and that there is a bijection from $A$ to $B$, $f : A \rightarrow B$. Since $f$ is a bijection we know that the function $f^{-1}$ is guaranteed to exist and is a bijection. Therefore $f^{-1} : B \rightarrow A$ and BRA, so $R$ is symmetric.

To verify that $R$ is transitive we must show that For any arbitrary sets $A,B,C$, IF ARB AND BRC, THEN ARC.
We will assume that ARB and BRC which means that there is a bijection $f$ from $A$ to $B$ and a bijection $g$ from $B$ to $C$. We will now create a bijection from $A$ to $C$. Consider the bijections $f$ and $g$ then the composite $(g \circ f) : A \rightarrow C$ is also a bijection since the composition of two bijections is always a bijection. Therefore ARC and $R$ is transitive. Since $R$ is reflexive, symmetric, and transitive, $R$ is an equivalence relation.

3) Which of the following relations are equivalence relations? Indicate any relation that is antisymmetric. Support your conclusions.
   a) $R = \{(x, y) \in R \times R : xy = 0\}$ Is symmetric only
   b) $R = \{(x, y) \in R \times R : xy \geq 0\}$ Is reflexive, symmetric, transitive
   c) $R = \{(x, y) \in R \times R : x \geq y\}$ Is reflexive, anti-symmetric, Transitive
   d) $R = \{(x, y) \in R \times R : x^2 + y^2 = 1\}$ Is symmetric
4) Let A be a set and P(A) be the power set of A. Let C and D be elements of P(A). Define the relation R to be CRD if \( C \subseteq D \). Show that R is a partial order relation.

**Proof:** To prove that R is a partial order we must show that R is reflexive, anti-symmetric, and transitive.

**To show that R is reflexive** we must show that FOR ALL sets B in P(A), BRB. This means that \( B \subseteq B \) which is true since every set is a subset of itself and R is reflexive.

**To show that R is anti-symmetric** we must show that for all sets C, D in P(A) IF CRD and DRC then C = D.

Let us assume that CRD and DRC, then we know that \( C \subseteq D \) and \( D \subseteq C \) and notice that this satisfies the definition for two sets to be equal, therefore \( C = D \) and R is anti-symmetric.

**To show that R is transitive** we must show that for all sets, IF CRD and DRE, then CRE. Let us assume that CRD and DRE so that \( C \subseteq D \) and \( D \subseteq E \) so \( C \subseteq E \) and R is transitive.

Since R is reflexive, anti-symmetric, and transitive R is a partial order on P(A)

5) Given a set A, for any a, b in A, aRb if and only if a divides b. Verify that R is a partial order on A.

(For this theorem we will assume a, b are natural numbers.)

**Proof:** To prove that R is a partial order we must show that R is reflexive, anti-symmetric, and transitive.

**To show that R is reflexive** we must show that FOR ALL natural numbers a, aRa. This means there must exist an integer q so that aq = a to satisfy the definition of divisible. The integer q = 1 will work for any natural numbers a and R is reflexive.

**To show that R is anti-symmetric** we must show that for all natural numbers, if aRb and bRa, then a = b. If we assume that aRb and bRa then by definition of divisible there exist integers k and p so that b = ka and a = pb. By substitution we have that \( b = k(pb) \) and \( b = (kp)b \). In order for this to be true kp must be equal to 1 and therefore \( k = 1 \) and \( p = 1 \) which will give us that a = b and b = a and R is anti-symmetric.

**To show that R is transitive** we must show that for all natural numbers, if aRb and bRc then aRc. Assuming that aRb and bRc, then there exist natural numbers m, f by the definition of divisible we have that \( b = am \) and \( c = bf \). Now consider \( c = (am)f = a(mf) \) so that \( mf = p \) which is a natural number and \( c = ap \). This satisfies the definition of divisible and aRc and R is transitive.

Since R is reflexive, anti-symmetric, and transitive R is a partial order.

6) If \( A = \{2, 3, 4, 6, 8, 9, 12, 18\} \) For all a, b in A, aRb iff a divides b.

Draw a Hasse Diagram representing R.

I have no way to illustrate this graph till I get back to the office.

2 will connect to 4 to 8 and 4 connects to 12
2 will connect to 6 to 12 and 6 connects to 18
3 will connect to 6 to 12 and to 6 and 18.
3 will connect to 9 and to 18.
7) Let A be a non empty finite set and B is a fixed subset of A. Define a relation R on The power set P(A) such that for any subsets C and D contained in P(A) CRD if 
\[ C \cap B = D \cap B. \]

a) Show that CRD is an equivalence relation of P(A).

1) Need to show that R is reflexive by showing for all subsets C that are elements of P(A) CRC and will satisfy \[ C \cap B = C \cap B. \] (How do you support this?)

2) Need to show that R is symmetric by showing that for sets C, D that If CRD then DRC. So we have that \[ C \cap B = D \cap B \] and must show that \[ D \cap B = C \cap B. \] How would justify this?

3) Need to show that R is transitive by showing that for sets C, D that If CRD and DRF, then CRF. We know that \[ C \cap B = D \cap B \] and \[ D \cap B = F \cap B \] And we need to show that \[ C \cap B = F \cap B. \] How would justify this?

b) For the particular case where \( A = \{ 1,2,3,4,5\} \) and \( B = \{1,2,5\} \) and \( C = \{ 2,4,5\} \), find all subsets D that relate with C and therefore will be in the equivalence class \([C]\). The sets in \([C]\) are all subsets in P(A) that when intersected with Set B will give \( \{2, 5\} \). You need to list the sets.