Problem 1: Given the following statements:

- p: Kirk is the captain,
- q: Spock is the science officer,
- r: Scotty is the engineer, and
- w: Bones is the doctor.

And also given the statement: \((-p \land -q) \rightarrow (r \lor w)\) is false determine if r, p, q, and w are true or false and justify your reasoning.

Solution: (10pts)

\((-p \land -q) \rightarrow (r \lor w)\) is given to be false this indicates that the necessary condition \(r \lor w\) is false and the sufficient condition \(-p \land -q\) is true.

Since \(r \lor w\) is false then r and w are each false since this disjunction.

In order for the conjunction \(-p \land -q\) the statements \(-p\) and \(-q\) must each be true.

Therefore p and q must each be false. So we have that p,q,r, and w are each false.

Problem 2: (14pts)

Use algebra of logic to prove the following statement. Justify each step.

\(~(-p \rightarrow [(p \land q) \lor -q]) = p \land q\)

Proof:

\(~(-p \rightarrow [(p \land q) \lor -q]) =\) given

\(~(p \lor [(p \land q) \lor -q]) =\) Equivalent form of implication

\(~((p \lor (p \land q)) \lor -q) =\) Associative Law

\(~(p \lor -q) =\) Absorption Law

\(~p \land q =\) Absorption Law

Therefore \(~(- p \rightarrow [(p \land q) \lor -q]) = p \land q\)
Problem 3: Express the following sentences in symbolic logic using multiple quantifiers. Specify domains for each variable used and define a predicate P(x,y)

a) Each of us has a favorite fast food restaurant.

b) Negate the symbolic logic in a) and then put into a natural English sentence.

Solution: (10pts)
Let P be the set of all people where x is an arbitrary person
Let F be the set of all fast food restaurants where y is an arbitrary fast food restaurant.
Let P(x,y) = x has favorite y
∀x ∈ P, ∃y ∈ F | P(x, y)

negate
∃x ∈ P | ∀y ∈ F, ~ P(x, y)

There is a person who does not have a favorite fast food restaurant.

Problem 4: Choosing only from the theorems given below, prove the following theorem. Do not try to reprove any of the listed theorems or use any definitions. Just quote the theorems you need to use in their entirety.

THEOREM: If b is a prime integer, then \( b(b + 1) + b^2 \) is odd for \( b > 2 \).

a) The sum of an even and odd integer is odd.

b) The product of two odd integers is odd.

c) Any prime number \( n > 2 \), is odd.

d) The product of two consecutive integers is even.

e) The product of two even integers is even.

f) The sum of two odd integers is even.

Proof: (14pts)
It has been proved that any prime number greater than 2 is an odd integer. So we know that since \( b \) is prime, it is odd. We also have that \( b^2 = (b)(b) \) is odd since the product of two odd numbers is odd. The product of \( b(b+1) \) is even since the product of two consecutive integers is even. The sum \( b(b+1) + b^2 \) is odd since the sum of an even integer and an odd integer is always odd.
**Problem 5:** Determine if the following argument is valid. First translate each statement into symbolic implication form and define your variables. Justify your reasoning in complete sentences.

Eva will pass this course if and only if she works hard. She will work hard or she will not graduate. But Eva will graduate. Therefore she will pass this course.

**Solution (14pts)**

Let $E$ be the statement: Eva will pass this course
Let $H$ be the statement: Eva will work hard
Let $G$ be the statement: Eva will graduate

Symbolic logic of given statements

$E \leftrightarrow H \equiv (E \rightarrow H) \land (H \rightarrow E)$ by definition of biconditional.

$H \lor \sim G \equiv \sim H \rightarrow \sim G \equiv G \rightarrow H$ by equivalent form of implication and the contrapositive.

$G$ 
$
\therefore E$

We are given that Eva did graduate and this is the sufficient condition for $G \rightarrow H$. Since this is given as a true implication the necessary must be true. So we now know that Eva studied hard. From the biconditional statement we know that $E \rightarrow H$ and $(H \rightarrow E)$. We now have that Eva studied hard is the sufficient condition for $H \rightarrow E$. Since the implication is given to be true then the necessary condition is true and we have that Eva passed the course.
Problem 6: (14pts) Proof by contrapositive

Theorem: If $8$ does not divide $n^2 - 1$, then $n \neq 4k + 1$

Proof by contrapositive: If $n = 4k + 1$ Then $8 \mid (n^2 - 1)$

Given that $n = 4k + 1$, consider the following:

$n^2 - 1 = (4k + 1)^2 - 1$

$= 16k^2 + 8k + 1 - 1$

$= 8(2k^2 + k) = 8p$ where $p = 2k^2 + k \in \mathbb{Z}$

By definition of divisible we have that $8 \mid (n^2 - 1)$

Since we have proved that the contrapositive is true we know that the original equivalent statement is also true and If $n = 4k + 1$ Then $8 \mid (n^2 - 1)$.

Problem 7: (14pts) Proof by contradiction.

Theorem: If $w$ is irrational then $\sqrt{w}$ is irrational for $w > 0$

Proof by contradiction.

Assume that $w$ is irrational and $\sqrt{w}$ is rational.

Assuming that $\sqrt{w}$ is rational then

$\sqrt{w} = \frac{a}{b}$ for non-zero integers $a$ and $b$ by definition of rational.

then $w = \frac{a^2}{b^2} = \frac{m}{n}$ where $m = a^2$ and $n = b^2$ are each integers.

So $w$ is rational by definition of rational, but this is a contradiction to our premises that $w$ is irrational.

Therefore $\sqrt{w}$ is irrational.
Problem 8: Use the Quotient Remainder Theorem to answer the following problem: The integers can be portioned into five distinct groups using \( n \mod 5 \). State the five groups and determine which group would contain the integer \( n = 128 \). Show all work.

Solution: (10pts) using \( n \mod 5 \) we can partition the integers into the following groups:

\[
\begin{align*}
&n = 5q \quad \text{or} \quad n = 5q + 1 \quad \text{or} \quad n = 5q + 2 \quad \text{or} \quad n = 5q + 3 \quad \text{or} \quad n = 5q + 4 \\
\end{align*}
\]

128 would fall the grouping \( 128 = 5q + 3 \) where \( q = 25 \)