Math 2534  Solution Test 2B  Spring 2015

Problem 1: (20pts) Use PMI to prove the following theorem and be clear where you use the inductive assumption. Justify all steps with complete sentences.

**Theorem:** For all natural numbers \( n \geq 5 \), \( 2n^2 < n! \)

**Proof:** To verify that the hypothesis is true for at least one value of \( n \), consider \( n = 5 \).

\[
2n^2 = 2(25) = 50 < 5! = 120
\]

Now assume that the hypothesis is true from \( n = 5 \) up to some arbitrary value \( k \) so that \( 2k^2 < k! \) and prove true for \( k + 1 \) by showing that \( 2(k+1)^2 < (k+1)! \).

Now consider the \( k + 1 \) term \( 2(k+1)^2 \).

\[
2(k+1)^2 = 2k^2 + 4k + 2 < k! + 4k + 2 \quad \text{by the inductive assumption.}
\]

So \( 2(k+1)^2 < k! + 4k + 2 < k! + 4k! + k! = 5k! \) since \( k! > k > 2 \)

and \( 2(k+1)^2 < 5k! < (k+1)k! = (k+1)! \) since \( k + 1 \geq 6 \).

Since we have assumed true up to \( k \) and proved true for \( k+1 \), the hypothesis is true for all \( n \geq 5 \).

**Alternate proof:** Let the \( k + 1 \) term be \( (k + 1)! \)

\[
(k + 1)! = k!(k + 1) > 2k^2(k + 1) \quad \text{by the inductive assumption}
\]

\[
(k+1)! > 2k^2(k + 1) = 2k^3 + 2k^2 > 2k^2 + 2k = k^2 + 2k + k^2 > k^2 + 2k + 1 = (k + 1)^2
\]

Problem 2: (18pts) Prove the following using Algebra of sets where \( A \) and \( B \) are nonempty sets

(Do not try to use elements in this type of proof and give reason for each step) Theorem:

\[ [A - (A \cap B)]^c = A^c \cup B \]

**Proof:**

\[
[A - (A \cap B)]^c = \quad \text{given}
\]

\[
[A \cap (A \cap B)^c]^c = \quad \text{by difference law}
\]

\[
[A^c \cup (A \cap B)^c] = \quad \text{by DeMorgan's law}
\]

\[
A^c \cup (A \cap B) = \quad \text{by double complement law}
\]

\[
(A^c \cup A) \cap (A^c \cup B) = \quad \text{by distributive law}
\]

\[
U \cap (A^c \cup B) = \quad \text{by the complement law}
\]

\[
A^c \cup B \quad \text{by identity law}
\]

Therefore \( [A - (A \cap B)]^c = A^c \cup B \)
Problem 3: (20pts) Use PMI to prove the following theorem and be clear where you use the inductive assumption. Justify all steps with complete sentences.

Theorem: Given the Fibonacci Sequence \( f_i = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}, \) for all natural numbers \( n > 2, \) then 
\[
f_n = 3f_{n-3} + 2f_{n-4}, \text{ for all natural numbers } n > 4.
\]

Proof: To verify that the hypothesis is true for at least one value of \( n, \) consider \( n = 5. \)

By definition of Fibonacci sequence:
\[
f_5 = f_4 + f_3 = 3 + 2 = 5
\]
\[
3f_{n-3} + 2f_{n-2} = 3f_2 + 2f_1 = 3(1) + 2(1) = 5
\]

Now consider \( n = 6, \)
\[
f_6 = f_5 + f_4 = 5 + 3 = 8
\]
\[
3f_{n-3} + 2f_{n-2} = 3f_3 + 2f_2 = 3(2) + 2(1) = 8
\]

Now assume that the hypothesis is true from \( n = 5 \) up to some arbitrary value \( k \) so that 
\[
f_k = 3f_{k-3} + 2f_{k-4}, \text{ and prove for } k + 1 \text{ by showing } f_{k+1} = 3f_{k-2} + 2f_{k-3}.
\]

For the body of the proof consider the \( k + 1 \) term \( f_{k+1}, \)
\[
f_{k+1} = f_k + f_{k-1} = (3f_{k-3} + 2f_{k-4}) + (3f_{k-4} + 2f_{k-5}) \text{ by the inductive assumption.}
\]
So \( f_{k+1} = (3f_{k-3} + 2f_{k-4}) + (3f_{k-4} + 2f_{k-5}) = 3(f_{k-3} + f_{k-4}) + 2(f_{k-4} + f_{k-5}) = 3f_{k-2} + 2f_{k-3} \)
by definition of the Fibonacci Sequence.
Since we have assumed true up to \( k \) and proved true for \( k+1, \) the hypothesis is true for all \( n > 4. \)

Problem 4: (15pts) Let a set of elements make up a Boolean Algebra \( B \) with operation \( \oplus \) with identity \( k \) and operation \( \odot \) with identity \( p. \) Let the complement of \( a \) be \( \bar{a}. \) Simplify the following expressions and justify your conclusions.

1) \( a \oplus p = p \) by universal bound law

2) \( a \odot \bar{a} = k \) by the inverse (complement law)

3) \( \bar{k} \oplus p = p \oplus p = p \) since identities are complements of each other and the idempotent law

4) \( a \oplus \bar{p} = a \oplus k = a \) by the identity law

5) \( (a \odot \bar{p}) \odot a = (a \odot k) \odot a = (k) \odot a = a \)
since identities are complements of each other
by the universal bound law
by the identity law
Problem 5: (12pts) Given the set \( A = \{ \emptyset, \{2\}, 2 \} \), determine if the following statements are true or false. If false, then correct the statement to be true.

a) \( \emptyset \in A \), false, but \( \emptyset \subseteq A \) is true.

b) \( \{\emptyset\} \subseteq P(A) \) true

c) \( \{2\} \in A \) true

d) \( \{\emptyset,\{\emptyset\}\} \subseteq A \) false, but \( \{\{\emptyset\}\} \subseteq A \) is true.

e) \( \{\{\emptyset\}\} \in P(A) \) true

f) \( \{\{2\}\} \subseteq A \) true

Problem 6: (15pt) Use proof by elements to verify that for all nonempty sets \( A, B, \) and \( D \)

Theorem: If \( A \subseteq B, D^c \subseteq B^c \) then \( D^c \subseteq A^c \). (Justify each step of proof)

Proof:

\( \forall x, x \in D^c \rightarrow x \in B^c \) since \( D^c \subseteq B^c \)

\( \rightarrow x \notin B \) by definition of complement

\( \rightarrow x \notin A \) since \( A \subseteq B \)

\( \rightarrow x \in A^c \) by definition of complement

By definition of containment \( D^c \subseteq A^c \)