Math 2534 Solution Homework 7 on PMI Spring 2015

**Theorem 1:** \( \forall n \in \mathbb{N}, \ 1+2+2^2+\ldots+2^{n-1} = 2^n - 1 \)

Proof by PMI: The hypothesis is true for at least one value on \( n \).

Consider the value \( n = 1 \) to see that \( 2^0 = 2^1 - 1 \) and it is clear that \( 1 = 1 \). In order to see the pattern clearly we will also consider \( n = 2 \) so that \( 2^0 + 2^2 = 2^2 - 1 \) which gives that \( 1 + 2 = 4 - 1 \) so that \( 3 = 3 \).

Assume that the hypothesis is true from \( n = 2 \) up to some arbitrary value \( k \) so that \( 1 + 2 + 2^2 + \ldots + 2^{k-1} = 2^k - 1 \) and prove true for \( k + 1 \) by showing that \( 1 + 2 + 2^2 + \ldots + 2^k = 2^{k+1} - 1 \).

For the body of the proof, consider the \( k + 1 \) term: \( 1 + 2 + 2^2 + \ldots + 2^k = \)

We now have that \( 1 + 2 + 2^2 + \ldots + 2^{k-1} + 2^k = \)

and by the inductive assumption we have that

\[
(1 + 2 + 2^2 + \ldots + 2^{k-1} ) + 2^k = \\
(2^k - 1) + 2^k = \\
2(2^k) - 1 = \\
2^{k+1} - 1
\]

I have assumed true up to \( k \) and proved true for \( k + 1 \), therefore the hypothesis is true for all natural numbers.

**Theorem 2:** \( \forall n \in \mathbb{N}, \ 3 \mid (4^n - 1) \)

Proof by PMI: The hypothesis is true for at least one value on \( n \) so consider \( n = 1 \).

In order to satisfy the definition of divisible we must show that there exist an integer \( q \) such that \( 3q = 4^1 - 1 \). So for \( q = 1 \) the definition is satisfied. Now consider \( n = 2 \). Does there exist an integer \( k \) such that \( 3q = 4^2 - 1 = 15 \). Clearly \( k = 5 \) will satisfy the definition of divisible.

Now assume that the hypothesis is true from \( n = 2 \) up to some arbitrary value \( k \) so that there exist some integer \( f \) such that \( 3f = 4^k - 1 \) and by definition of divisible \( 3 \mid (4^k - 1) \). In order to prove true for \( k + 1 \) we must show there exist an integer \( h \) so that \( 3h = 4^{k+1} - 1 \) and therefore

\( 3 \mid (4^{k+1} - 1) \). For the body of the proof consider the \( k + 1 \) term.

\[
4^{k+1} - 1 = (4^k)4 - 1 = (4^k)4 - 1 = (4^k - 1) + 3(4^k) \]

By the inductive assumption we have that

\[
(4^k - 1) + 3(4^k) = 3f + 3(4^k) = 3f + 3m \]

where \( f \) and \( m = 4^k \) are integers. Therefore we have the final results that \( 4^{k+1} - 1 = 3f + 3m = 3h \) for integer \( h = f + m \).

By definition of divisible \( 3 \mid (4^{k+1} - 1) \).

I have assumed true up to \( k \) and proved true for \( k + 1 \), therefore the hypothesis is true for all natural numbers.
**Theorem 3:** \( \forall n \in N \geq 5, \quad (n + 1)! > 2^{n+3} \)

**Proof by PMI:** The hypothesis is true for at least one value on \( n \) so consider \( n = 5 \). We have that \((5+1)! > 2^{5+3}\) which will give the following results, \((6)! > 2^8\) and therefore \(720 > 256\). Now assume that the hypothesis is true from \( n = 5 \) up to some arbitrary value \( k \) so that 
\[(k + 1)! > 2^{k+3}\]. Now prove true for \( k + 1 \) by showing \((k + 2)! > 2^{k+4}\).

For the body of the proof consider the \( k + 1 \) term \((k + 2)!\)
\[(k + 2)! = (k + 2)(k + 1)! \quad \text{and by the inductive assumption}
(k + 2)! = (k + 2)(k + 1)! > (k + 2)2^{k+3} \quad \text{Since } k > 4, \text{ } k + 2 > 2 \text{ and we have that}
(k + 2)! = (k + 2)(k + 1)! > (k + 2)2^{k+3} > 2(2^{k+3}) = 2^{k+4}\]
Since we assumed true up to \( k \) and proved true for \( k + 1 \), therefore the hypothesis is true for all natural numbers greater than 4.

**Theorem 4:** \( \forall n \in N \geq 4, \quad 2n + 3 \leq 2^n \)

**Proof by PMI:** The hypothesis is true for at least one value on \( n \) so consider \( n = 4 \). We have that \(2(4) + 3 = 11 \leq 2^4 = 16\).

Now assume that the hypothesis is true from \( n = 4 \) up to some arbitrary value \( k \) so that \(2k+3 \leq 2^k\). Now prove true for \( k + 1 \) by showing \(2(k+1)+3 \leq 2^{k+1}\).

In the body of the proof consider the \( k + 1 \) term \(2(k+1) + 3\).
Now we see that \(2(k+1) + 3 = 2k + 2 + 3 = (2k + 3) + 2 \leq 2^k + 2 \) by the inductive assumption. Since \( k > 3 \) then \(2 < 2^k\) and we have that \(2^k + 2 \leq 2^k + 2^k = 2(2^k) = 2^{k+1}\).
So \(2(k+1) + 3 \leq 2^{k+1}\).
Since we assumed true up to \( k \) and proved true for \( k + 1 \), therefore the hypothesis is true for all natural numbers greater than 3.

**Theorem 5:** If \( f(x) = x^2e^x \), then \( f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)], \forall n \in N, x \in R \)

**Proof by PMI:** The hypothesis is true for at least one value on \( n \) so consider \( n = 1 \).
For \( n = 1 \) we have that \( f(x) = x^2e^x \), \( f'(x) = 2xe^x + x^2e^x = e^x[x^2 + 2(1)x + (1)(1-1)] \).

Also consider \( n = 2 \) to have \( f''(x) = x^2e^x + 4xe^x + 2e^x = e^x[x^2 + 2(2)x + (2)(2-1)] \)

Now assume that the hypothesis is true from \( n = 2 \) up to some arbitrary value \( k \) so that \( f^{(k)}(x) = e^x[x^2 + 2kx + k(k-1)] \). Now prove true for \( k + 1 \) by showing \( f^{(k+1)}(x) = e^x[x^2 + 2(k+1)x + (k+1)(k)] \).

For the body of the proof consider the \( k + 1 \) term \( f^{(k+1)}(x) \). So we have that
\[f^{(k+1)}(x) = \left( f^{(k)}(x) \right)' = \frac{df^{(k)}}{dx}(e^x[x^2 + 2kx + k(k-1)]) = e^x(x^2 + 2kx + k(k-1)) + e^x(2x + 2k)\]
\[= e^x(x^2 + 2(k+1)x + (k+1)(k))\]
I have assumed true up to $k$ and proved true for $k + 1$, therefore the hypothesis is true for all natural numbers.