Math 2534 Solution Homework 4 on Proofs

Be precise on domain used and definitions. The universal quantifier must be clearly implied for valid theorems and the existential quantifier clearly implied for counter-examples. Define your variables clearly and use complete sentences in your write-ups.

Problem 1: Use direct proof using definitions only or give counter-example

Theorem: For all integers, if a is even and b is odd then \( a^2 - 3b \) is odd.

Proof: Given that a is even, then by definition of even there exist an integer k so that \( a = 2k \).
Since b is odd then by definition of odd there exist an integer p so that \( b = 2p + 1 \).
Now consider \( a^2 - 3b = (2k)^2 - 3(2p + 1) = 4k^2 - 6p - 3 = 4k^2 - 6p - 4 + 1 \)
\[ = 2(2k^2 - 3p - 2) + 1 = 2m + 1 \quad \text{where} \ m = 2k^2 - 3p - 2 \in \mathbb{Z}. \]
Therefore by definition of odd we have proved that \( a^2 - 3b = 2m + 1 \) is odd.

Problem 2: Use direct proof using definitions only or give counter-example

Theorem: For all natural numbers, If n is odd then n is prime.

Counter-example: Consider the odd natural number \( n = 21 \). The natural number \( n = 21 \) can be divided by 21, 3, 7, 1 and does not satisfy the definition of prime.

Problem 3: Use direct proof using definitions only or give counter-example

Theorem: If a, b and c are natural numbers and \( a \mid b \) and \( a \mid c \) then \( a \mid (3c + b) \)

Proof: Since we are given that a divides b evenly and a divides c evenly, then by definition of divisible there exist integers k and q such that \( ak = b \) and \( aq = c \).
Now consider \( 3c + b = 3(aq) + ak = a(3q + k) = am \) where \( m = 3q + k \) is an integer. By definition of divisible we have that a divides \( 3c + b \) evenly.