Math 2534  Solutions  Homework 4

Instructions: Prove or give a counterexample. Justify all assertions made.

**Problem 1: Direct proof:**
Theorem: For all natural numbers, if a and b are each prime numbers greater than 2, then a + b is even.
Proof: Given a and b are each prime numbers, we have that a and b are odd since all prime numbers greater than 2 are odd. Therefore we have that a + b are even since we know that the sum of two odd numbers are always odd.

**Problem 2: Direct Proof:**
Theorem: If a, b and c are natural numbers and \( a | c \) and \( b | d \) then \( ab | cd \)
Proof: Given that \( a | c \) and \( b | d \) we have that \( aq = c \) and \( bk = d \) for integers q and k by definition of divisible. Now consider
\[
aq = c
\]
\[
aqd = cd \quad \text{multiplying through by} \ d
\]
\[
aq(bk) = cd \quad \text{since} \ bk = d
\]
\[
ab(qk) = cd \quad \text{by communtative and associative laws}
\]
\[
ab(m) = cd \quad \text{where} \ m = qk \text{ is an integer}
\]
so \( ab | cd \) by definition of divisible.

**Problem 3: Indirect Proof by contrapositive**
Theorem: If \( n^3 + 5 \) is odd then n is even for all natural numbers.
**Proof by contrapositive:** If n is odd then \( n^3 + 5 \) is even
We know that n is odd so \((n)(n) = n^2\) is also odd since the product of two odd numbers is odd. We now have that \( n^3 = n^2n \) is odd by the same theorem. We also have that \( n^3 + 5 \) is even since we know that the sum of two odd numbers is always even. We have proven the contrapositive is true and so the equivalent original statement is also true.

**ALTERNATE PROOF by contrapositive.**
By definition of odd we have that n = 2p + 1 for integer p. Now consider the following
\[
n^3 + 5 = (2p + 1)^3 + 5 = 8p^3 + 12p^2 + 6p + 6 = 2(4p^3 + 6p^2 + 3p + 3) = 2m
\]
where \( m = 4p^3 + 6p^2 + 3p + 3 \) is an integer
Therefore \( n^3 + 5 \) is even by definition by even. Since the contrapositive is true the equivalent statement is also true.
Problem 4: **Indirect Proof by contradiction**

Theorem: If \( m \) and \( n \) are integers and the product \( mn \) is even then \( m \) is even or \( n \) is even.

Proof by contradiction;

Assume there are some integers \( m \) and \( n \) so that \( mn \) is even and \( m \) is odd and \( n \) is odd.

Assuming that \( m \) and \( n \) are both we will have that \( mn \) is odd since the product of two odd numbers is always odd. BUT this is a contradiction since \( mn \) is given to be even. Therefore \( m \) or \( n \) is even.

(ALTERNATE PROOF would use the definitions of even and odd)

Problem 5: **Prove or disprove.**

Theorem: For all natural numbers, If \( n \) is odd then \( n \) is prime.

Consider \( n = 35 \), It is odd but it is composite and not prime since \( 35 = (7)(5) \).