Math 2534 Solution Homework 10 on sets and Boolean Algebra

Show all work.

**Problem 1:**
Use proof by elements to prove the following:

**Theorem:** For all sets A, B \( P(A) \cup P(B) \subseteq P(A \cup B) \)

**Proof:**
\[ \forall x, x \in P(A) \cup P(B) \rightarrow x \subseteq A \vee x \subseteq B \quad \text{by definition of powerset and of union} \]
\[ \rightarrow x \subseteq (A \cup B) \quad \text{by definition of union} \]
\[ \rightarrow x \in P(A \cup B) \quad \text{by definition powerset} \]
therefore \( P(A) \cup P(B) \subseteq P(A \cup B) \) by definition of containment

**Problem 2:** Use set Algebra to prove the following and justify each step.

a) **Theorem:** For any sets A, B, C such that \( (B - A) \cap (C - A) = (B \cup C) - A \)

**Proof:**
\( (B - A) \cap (C - A) = \) given
\( (B \cap A^c) \cap (C \cap A^c) = \) by the difference law
\( A^c \cap (B \cup C) = \) by the distributive law
\( (B \cup C) \cap A^c = \) by the commutative law
\( (B \cup C) - A = \) by the difference law
Therefore \( (B - A) \cap (C - A) = (B \cup C) - A \)

b) **Theorem:** For any sets A, B, C, \( [A^c \cup (B - A)]^c \cap A = A \)

**Proof:**
\( [A^c \cup (B - A)]^c \cap A = \) given
\( [A^c \cup (B \cap A^c)]^c \cap A = \) by the difference law
\( A^c \cap (B \cap A^c)^c \cap A = \) by the DeMorgan's law
\( A \cap (B \cap A^c)^c \cap A = \) by the double complement law
\( (A \cap A) \cap (B \cap A^c) = \) by the commutative and associative law
\( A \cap (B \cap A^c) = \) by the idempotent law
\( A \cap (B^c \cup A^c) = \) by DeMorgan's law
\( A \cap (B^c \cup A) = \) by the double complement law
\( A = \) by absorption law
Therefore \( [A^c \cup (B - A)]^c \cap A = A \)

**Problem 3:** Given elements a, b in the Boolean algebra B with operations \( \oplus \) and \( \odot \) where m is the identity for \( \oplus \) and p is the identity for \( \odot \), Justify each step of the proof below. The inverse of any element a is \( a' \).
**Theorem:** For a, b in B, \((a \boxtimes b) \ominus (b \boxtimes b) = m\)

Proof:
- \((a \boxtimes b) \ominus (b \boxtimes b) = \text{given}\)
- \((a \boxtimes b) \ominus b = \text{indempotent law}\)
- \((a' \odot b') \ominus b = \text{DeMorgan’s law}\)
- \(a' \odot (b' \ominus b) = \text{associative law}\)
- \(a' \odot m = \text{complement law}\)
- \(m = \text{universal bound}\)

**Problem 4:** Given elements a, b in the Boolean algebra B with operations \(\otimes\) and \(\odot\) where k is the identity for \(\otimes\) and h is the identity for \(\odot\). Let \(\overline{b}\) be the complement of b. Justify each step of the proof below.

**Theorem:** For a, b in B, \([b \otimes (\overline{b} \odot a)] \odot (\overline{b} \odot a) = \overline{a}\)

Proof:
- \([b \otimes (\overline{b} \odot a)] \odot (\overline{b} \odot a) = \text{given}\)
- \([b \otimes (\overline{b} \odot a)] \odot (\overline{b} \odot a) = \text{DeMorgan’s law and double compliment law}\)
- \([b \otimes (\overline{b} \odot a)] \odot (\overline{b} \odot a) = \text{by the associative law}\)
- \([b \otimes \overline{a}] \odot (\overline{b} \odot a) = \text{by the idempotent law}\)
- \([b \otimes \overline{a}] \odot (\overline{b} \odot a) = \text{by DeMorgan's law}\)
- \((b \otimes \overline{b}) \otimes \overline{a} = \text{by the distributive law}\)
- \(k \otimes \overline{a} = \text{by the complement law}\)
- \(\overline{a} = \text{by the identity law}\)

Therefore \([b \otimes (\overline{b} \odot a)] \odot (\overline{b} \odot a) = \overline{a}\)