**Math 2534  Solution  Homework 4**

**Problem 1: Direct proof:**
Theorem: For all integers, if $a$ is even and $b$ is odd then $a^2 - 3b$ is odd.
Proof: Given that $a$ is an even integer then we have $a^2 = aa$ is even, since the product of two even numbers is even. Given that $3$ is an odd integer and $b$ is an odd integer, we have that the product $3b$ is also odd since the product of two odd integers is always odd. Now we can claim that $a^2 - 3b$ is odd since the difference of an even integer minus an odd integer is odd.

**Problem 2: Direct Proof:**
Theorem: If $a$, $b$ and $c$ are natural numbers and $a | b$ and $a | c$ then $a | (b - 2c)$
Proof: Since $a | b$ and $a | c$, and by definition of divisible, we have that $b = ak$ and $c = ap$ for some integers $k$ and $p$. Now consider $b - 2c = ak - 2(ap) = a(k - 2p) = a(m)$ where $m = k - 2p$ is an integer. Therefore by definition of divisible we have that $a | (b - 2c)$.

**Problem 3: Indirect Proof by contrapositive**
Theorem: If $n^2$ is odd then $n$ is odd for all natural numbers.
Proof by contrapositive: Restatement: If $n$ is even then $n^2$ is even.
Since we are given that $n$ is even, then $n^2 = nn$ is also even because the product of two even integers is always even. We have shown the contrapositive is true, therefore the equivalent original statement is also true and If $n^2$ is odd then $n$ is odd for all natural numbers.

**Problem 4: Indirect Proof by contradiction**
Theorem: For all non zero rational numbers, the product of a rational number and an irrational number is always irrational.
Proof by contrapositive: Assume that the product of a rational number and irrational number is rational.
Let $r$ be a rational number so that by definition $r = \frac{a}{b}$ were $a$ and $b$ are non zero integers. Let $w$ be an irrational number so that the sum $r + w$ is rational. We can now represent $r + w = \frac{c}{d}$ for non zero integers $c$ and $d$. This will give the following calculations where $r + w = \frac{c}{d}$.

\[
\frac{a}{b} = \frac{c}{d} \quad \text{and} \quad w = \left( \frac{c}{d} \right) \left( \frac{b}{a} \right) = \frac{cb}{da}, \quad \text{where da \neq 0}\]

This will give us that $w$ is rational by definition of rational. This contradicts that $w$ is given to be irrational. Therefore the original statement is true and the product is irrational.
Problem 5: Your choice
Theorem: For all natural numbers, if $n$ is odd then $n$ is prime.
Counter Example: let $n = 25$ which is odd but not prime.

Problem 6: Your choice
Theorem: For all natural numbers, if $a$ and $b$ are each prime numbers, then $a + b$ is even.
Counter Example: Let $a = 2$ and $b = 7$, then $2 + 7 = 9$ which is not even.