Curves and positroids in the Grassmannian

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Gromov-Witten varieties

- $X = \text{Gr}(p, m)$ - transitive action of $\text{GL}_m(\mathbb{C})$ acts transitively on $X$. Take $B^+, B^-$ the upper/lower triangular matrices in $\text{GL}_m(\mathbb{C})$.
- Closures of $B^+$-orbits $\leftrightarrow$ Schubert varieties $\Omega_\lambda$
- Closures of $B^-$-orbits $\leftrightarrow$ opposite Schubert varieties $\Omega_{\mu}^{opp}$.
- Fix $d \geq 0$. $\overline{\mathcal{M}}_{0,3}(X, d)$ compactifies the space of maps $f : (\mathbb{P}^1, pt_1, pt_2, pt_3) \to X$ such that $f_*[\mathbb{P}^1] = d[\text{line}]$.
- evaluation maps: $\text{ev}_i : \overline{\mathcal{M}}_{0,3}(X, d) \to X$ given by $\text{ev}_i(f) = f(pt_i)$. 
Gromov-Witten varieties

**Definition:** Gromov-Witten variety

\[ GW_d(\lambda, \mu) = ev_1^{-1} \Omega_\lambda \cap ev_2^{-1} \Omega_\mu \]

**Theorem (BCMP)**

\( GW_d(\lambda, \mu) \) is either empty or it is irreducible and unirational, with rational singularities. (This holds for any \( G/P \).)

Y is **unirational**: \( \exists F : \mathbb{P}^N \rightarrow Y \) dominant.

Y has **rational singularities** if \( \exists \) desingularization \( F : Z \rightarrow Y \) so that \( F_* \mathcal{O}_Z = \mathcal{O}_Y \) and \( R^i F_* \mathcal{O}_Z = 0, i > 0. \)

**Definition.**

\[ \Gamma_d(\lambda, \mu) = ev_3(GW_d(\lambda, \mu)) \]

- This is a subvariety of the Grassmannian;
- It is the union of all rational curves of degree \( d \) joining \( \Omega_\lambda \) and \( \Omega_\mu^{opp} \).
\[ \Gamma_d(\lambda, \emptyset) \] is the union of all rational curves of degree \( d \) passing through \( \Omega_\lambda \).

**Proposition.** [Carrell-Peterson, Fulton-Woodward] \( \Gamma_d(\lambda, \emptyset) = \Omega_\lambda[-d] \)
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\( \lambda[-1] \)
Proposition. [Carrell-Peterson, Fulton-Woodward] $\Gamma_d(\lambda, \emptyset) = \Omega_{\lambda[-d]}$

$\Gamma_d(\lambda, \emptyset)$ is the union of all rational curves of degree $d$ passing through $\Omega_{\lambda}$. 

$\lambda[-2]$
Definition. [Lusztig, Rietsch, Knutson-Lam-Speyer] Let $R$ be a Richardson variety in the full flag manifold $Fl(n)$. A **positroid** is a projection $\pi(R)$, where $\pi : Fl(m) \to Gr(p, m)$ is the projection.

Theorem. [Knutson-Lam-Speyer] Positroid varieties are normal and have rational singularities.
Gromov-Witten positroids

\[ \{ K^{p-d} \subset V \subset S^{p+d} \} \xrightarrow{\pi_1} \text{Gr}(p, m) = \{ V \} \]
\[ \downarrow \pi_2 \]
\[ Y = \{ K^{p-d} \subset S^{p+d} \subset \mathbb{C}^m \} \]

\[ \Omega_\lambda \subset \text{Gr}(p, m) - \text{Schubert variety} \]
\[ Y_\lambda = \pi_2(\pi_1^{-1}\Omega_\lambda) \]

\[ R_d(\lambda, \mu) = \pi_2^{-1}(Y_\lambda \cap Y_\mu^{opp}) \subset \text{Fl}(p - d, p, p + d; m) \]

\( R_d(\lambda, \mu) \) is a Richardson variety and \( \pi_1(R_d(\lambda, \mu)) \) is a GW positroid.
Gromov-Witten positroids

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\{K^{p-d} \subset V \subset S^{p+d}\} \xrightarrow{\pi_1} \text{Gr}(p, m) = \{V\} \\
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\[\Omega_\lambda \subset \text{Gr}(p, m) - \text{Schubert variety}\]

\[Y_\lambda = \pi_2(\pi_1^{-1}\Omega_\lambda)\]

\[R_d(\lambda, \mu) = \pi_2^{-1}(Y_\lambda \cap Y_\mu^{\text{opp}}) \subset \text{Fl}(p - d, p, p + d; m).\]

\(R_d(\lambda, \mu)\) is a Richardson variety and \(\pi_1(R_d(\lambda, \mu))\) is a GW positroid.

**Condition DIM.** Say \(\pi_1(R_d(\lambda, \mu))\) satisfies condition DIM if

\[\dim R_d(\lambda, \mu) = \dim \pi_1(R_d(\lambda, \mu)).\]
**Definition of small quantum cohomology**

\[ \text{Gr}(p, m) = \{ V \subset \mathbb{C}^m : \dim V = p \} \] - Grassmannian of \( p \)-planes in \( \mathbb{C}^m \).

- \( \text{QH}^*(\text{Gr}(p, m)) \) is a graded \( \mathbb{Z}[q] \)-algebra, where \( \deg q = m \).
- \( \text{QH}^*(X) \) has a \( \mathbb{Z}[q] \)-basis \( \{ [\Omega_\lambda] \} \) - the Schubert classes.

Multiplication:

\[
[\Omega_\lambda] \ast [\Omega_\mu] = \sum_{d \geq 0} \sum_{\nu} q^d \langle \Omega_\lambda, \Omega_\mu, \Omega_\nu^\vee \rangle_d [\Omega_\nu].
\]

- \( \langle \Omega_\lambda, \Omega_\mu, \Omega_\nu^\vee \rangle_d \) is the 3 point, genus 0 GW invariant.
- \( \langle \Omega_\lambda, \Omega_\mu, \Omega_\nu^\vee \rangle_d \) equals the number of rational curves in \( X \), passing through translates of Schubert varieties \( \Omega_\lambda, \Omega_\mu \) and \( \Omega_\nu^\vee \).
Theorem. [Buch-Kresch-Tamvakis, Postnikov, Knutson-Lam-Speyer] Assume condition DIM holds. Then:

1. $\Gamma_d(\lambda) = \pi_1(R_d(\lambda, \mu))$ so it is a positroid GW variety.
2. The class of $\Gamma_d(\lambda, \mu) \in H^*(\text{Gr}(p, m))$ is

$$[\Gamma_d(\lambda, \mu)] = \sum \langle [\Omega_\lambda], [\Omega_\mu], [\Omega_\nu]^\vee \rangle_d [\Omega_\nu]$$

Moreover, condition DIM holds $\iff$ $q^d$ appears in $[\Omega_\lambda] \ast [\Omega_\mu] \iff \mu^\vee / d / \lambda$ is toric.
K-theory class of GW positroids

Theorem (B-C-M-P)

1. $\Gamma_d(\lambda, \mu) = \pi_1(R_d(\lambda, \mu))$ is always a positroid GW variety.
2. The K-theory class of $\Gamma_d(\lambda, \mu)$ is given in terms of K-theoretic GW invariants:

$$[\mathcal{O}_{\Gamma_d(\lambda, \mu)}] = \sum \langle [\mathcal{O}_{\Omega_{\lambda}}], [\mathcal{O}_{\Omega_{\mu}}], [\mathcal{O}_{\Omega_{\nu}}]^\vee \rangle_d [\mathcal{O}_{\Omega_{\nu}}]$$

Example. $X = \text{Gr}(2, 4), d = 1, \lambda = \mu = (2)$. It is known that $[\Omega_{(2)}] * [\Omega_{(2)}] = [\mathcal{O}_{(2,2)}]$. No $q^1$ power, so DIM does not hold!
K-theory class of GW positroids

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$$[O_{\Gamma_d(\lambda, \mu)}] = \sum \langle [O_{\Omega_{\lambda}}], [O_{\Omega_{\mu}}], [O_{\Omega_{\nu}}]^\vee \rangle_d [O_{\Omega_{\nu}}]$$

Example. $X = \text{Gr}(2, 4), d = 1, \lambda = \mu = (2)$. It is known that $[\Omega_{(2)}] * [\Omega_{(2)}] = [O_{(2, 2)}]$. No $q^1$ power, so DIM does not hold!

$$\Gamma_1((2), (2)) = \Gamma_1(pt, \emptyset) = \Omega_{(1)}$$

This implies:

- $\langle [O_{(2)}], [O_{(2)}], [O_{\nu}]^\vee \rangle_1 = 0$ if $\nu \neq (1)$;
- $\langle [O_{(2)}], [O_{(2)}], [O_{(1)}]^\vee \rangle_1 = 1$. 
Rational neighborhoods of GW positroids

In the study of $QK(\text{Gr}(p, m))$ the following variety arises naturally:

$$\Gamma_{d_1}(\lambda, \mu) \subset \Gamma_{d_1,d_2}(\lambda, \mu) \subset \text{Gr}(p, m)$$

the union of all rational curves of degree $d_2$ passing through $\Gamma_{d_1}(\lambda, \mu)$.
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$$

the union of all rational curves of degree $d_2$ passing through $\Gamma_{d_1}(\lambda, \mu)$.

Example. $\Gamma_{0,1}(\lambda, \mu) = \text{union of lines through } \Omega_\lambda \cap \Omega_{\mu}^{opp}$. 
Open questions

1. Find geometric properties for $\Gamma_{d_1,d_2}(\lambda, \mu)$. Given a conjectural formula for K-class of $\Gamma_{d_1,d_2}(\lambda, \mu)$ we expect that this variety is normal and it has rational singularities.

2. Examples show:

$$\text{GW positroids } \subsetneq \{ \text{positroids} \}$$

$$\{ \text{positroids} \} \text{ almost equal } \{ \Gamma_{d_1,d_2}(\lambda, \mu) \}$$

Is there a general statement?

3. Other homogeneous spaces $G/P$? Any connections to Lusztig stratification?
Thank you!