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Classical Rings of Quotients Lamplighter Group

Joint work with

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Ore Condition

Definition. R a ring. Then $s \in R$ is a nonzero divisor means $sr \neq 0 \neq rs$ whenever $0 \neq r \in R$.
Definition. R a ring and S the set of nonzero divisors of R. Then R satisfies the right Ore condition means given $r \in R$ and $s \in S$, there exist $r_1 \in R$ and $s_1 \in S$ such that $rs_1 = sr_1$.

For group rings kG, right and left Ore condition are equivalent.

When R satisfies the Ore condition, we can form the Ore localization RS^{-1}, which consists of elements rs^{-1} with $r \in R$ and s a nonzero divisor. This ring has the property that every element is either invertible or a zero divisor, and is often called a classical ring of quotients for R. Also RS^{-1} is flat over R.

Proposition. *Let k be a field and G the non-abelian free group of rank 2. Then kG does not have a classical ring of quotients, even though it is embeddable in a division ring.*

Theorem (Kropholler, Linnell, Moody). *Let k be a field and G an elementary amenable group. Assume that the orders of finite subgroups are bounded. Then kG has a classical ring of quotients.*

Proposition. *Let k be a field and G a locally finite group. Then kG is its own classical ring of quotients.*

Theorem (Tamari). *Let k be a field and G an amenable group. Assume that kG is a domain. Then kG has a classical ring of quotients.*

Recall:
solvable-by-finite $\subset$ elementary amenable $\subset$ amenable $\subset$ no nonabelian free subgroup

Problems. *Let k be a field. For which groups G does kG have a classical ring of quotients?*

Lamplighter Group

Definition. The lamplighter group L is the Wreath product $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$. Thus the base group B of L is $\bigoplus_{-\infty}^{\infty} \mathbb{Z}/2\mathbb{Z}$ and $L/B \cong \mathbb{Z}$.

L is solvable, but its finite subgroups do not have bounded order.

Theorem 1. *Let k be a field. Then kL does not have a classical ring of quotients.*

Sketch Proof case $k = \mathbb{C}$

Lemma 2. *Let R be a subring of the ring S and let P be a projective R-module. If $P \otimes_R S$ is finitely generated as an S-module, then P is finitely generated.*

Lemma 3. *Let Q be a ring containing $\mathbb{C}L$ such that $\mathbb{C} \otimes_{\mathbb{C}L} Q = 0$. Then* $\text{Tor}^{\mathbb{C}L}(\mathbb{C}, Q) \neq 0$.

Proof. Let Δ denote the augmentation ideal of $\mathbb{C}L$; that the $\mathbb{C}$-subspace of $\mathbb{C}L$ with basis $\{g-1 \mid g \in L\}$. There are short exact sequences

$$0 \to \Delta \to \mathbb{C}L \to \mathbb{C} \to 0$$
$$0 \to P \to \mathbb{C}L^2 \to \Delta \to 0.$$

Note P is a projective $\mathbb{C}L$-module because $\mathbb{C}L$ has cohomological dimension 2.

If $\text{Tor}^{\mathbb{C}L}(\mathbb{C}, Q) = 0$, then have exact sequence

$$0 \to \Delta \otimes_{\mathbb{C}L} Q \to \mathbb{C}L \otimes_{\mathbb{C}L} Q \to \mathbb{C} \otimes_{\mathbb{C}L} Q = 0.$$

Homological dimension of $\mathbb{C}L$ is 1, so

$$0 = \text{Tor}_2^{\mathbb{C}L}(\mathbb{C}, Q) = \text{Tor}_1^{\mathbb{C}L}(\Delta, Q)$$

hence exact sequence

$$0 \to P \otimes_{\mathbb{C}L} Q \to Q^2 \to Q \to 0$$

Since this sequence splits, $P \otimes_{\mathbb{C}L} Q$ is a finitely generated Q-module. Therefore P is finitely generated by Lemma 2.

This contradicts the fact that L is not almost finitely presented over $\mathbb{C}$. If

$$1 \to R \to F \to L \to 1$$

is a presentation of L with F free of rank 2, then $R/R' \otimes_{\mathbb{Z}} \mathbb{C}$ is not finitely generated as a $\mathbb{C}L$-module. $\square$

Proof of Theorem 1. Suppose $\mathbb{C}L$ has a classical quotient ring Q. Since $L/B \cong \mathbb{Z}$, there exists $x \in L$ with infinite order. Then $1 - x$ is a nonzero divisor in $\mathbb{C}L$, hence $1 - x$ is invertible in Q. Thus $\mathbb{C} \otimes_{\mathbb{C}L} Q = 0$. Therefore $\mathrm{Tor}^{\mathbb{C}L}(\mathbb{C}, Q) \neq 0$ by Lemma 3. This contradicts Q flat over $\mathbb{C}L$. □

Unbounded Operators

Definition. $L^2(G)$: Hilbert space with Hilbert basis G

$\mathcal{B}(H)$: bounded linear operators on Hilbert space H

$\mathcal{N}(G)$: weak closure of $\mathbb{C}G$ in $\mathcal{B}(L^2(G))$

$\mathcal{U}(G)$: unbounded operators affiliated to $\mathcal{N}(G)$

$\mathcal{D}(G)$: division closure of $\mathbb{C}G$ in $\mathcal{U}(G)$.

$$\begin{array}{ccccc} \mathcal{D}(G) & & \subset & & \mathcal{U}(G) \\ \cup & & & & \cup \\ \mathbb{C}G & \subset & \mathcal{N}(G) & \subset & L^2(G) \end{array}$$

Inclusions proper if G infinite

Remark. $\mathcal{U}(G)$ is a von Neumann regular ring.

There is a dimension function on $\mathcal{U}(G)$-modules with $\dim \mathcal{U}(G) = 1$.

Special case: if M is a finitely generated right ideal of $\mathcal{U}(G)$, then $M = e\mathcal{U}(G)$ for unique projection $e \in L^2(G)$. Then $\dim M = e_1$.

$\mathcal{U}(G)$ is dimension flat over $\mathbb{C}G \Leftrightarrow G$ is amenable

$\mathcal{D}(G)$ is the smallest subring of $\mathcal{U}(G)$ containing $\mathbb{C}G$ which is closed under taking inverses.

If G is amenable, then every element of $\mathcal{D}(G)$ is either invertible or a zero divisor in $\mathcal{D}(G)$.

Proof of Theorem 1 shows neither $\mathcal{D}(L)$ nor $\mathcal{U}(L)$ flat over $\mathbb{C}L$, even though L is amenable.

Problems. *Is $\mathcal{D}(L)$ von Neumann regular?*

If $\alpha \in \mathbb{C}L$, then we know

$$\ker \beta \mapsto \alpha\beta \colon \mathcal{U}(L) \to \mathcal{U}(L) = e\mathcal{U}(L)$$

for some projection $e \in \mathcal{U}(L)$. Is $e \in \mathcal{D}(L)$?

Is $\dim e\mathcal{U}(L) \in \mathbb{Q}$*? (Atiyah)*