First Homework Solutions

1. **1.1.9(b) on page 22** Let \( G = \{a + b \sqrt{2} \in \mathbb{R} \mid a, b \in \mathbb{Q}\} \). Prove that the nonzero elements of \( G \) form a group under multiplication.

If \( x, y \in G \) then we may write \( x = a + b \sqrt{2} \) and \( y = c + d \sqrt{2} \) where \( a, b, c, d \in \mathbb{Q} \), and then \( xy = (ac + 2bd) + (ad + bc) \sqrt{2} \). Also if \( x, y \neq 0 \), then \( xy \neq 0 \). It follows that multiplication is a binary operation on \( G \setminus 0 \). Also multiplication is associative and the identity is 1. Finally we need to check for inverses. If \( a + b \sqrt{2} \in G \setminus 0 \), then the inverse will be \( 1/(a + b \sqrt{2}) \); the only problem we might have is that this is not obviously in \( G \setminus 0 \). However by multiplying top and bottom by \( a - b \sqrt{2} \), this is

\[
\frac{a}{a^2 - 2b^2} - \frac{b \sqrt{2}}{a^2 - 2b^2}.
\]

Since \( a^2 - 2b^2 \) is a nonzero rational number, it is now clear that the inverse is in \( G \setminus 0 \) (if \( a^2 - 2b^2 = 0 \), then \( \sqrt{2} \in \mathbb{Q} \)) and the result follows.

2. **1.1.14 on page 22** Find the orders of the following elements of the multiplicative group \((\mathbb{Z}/36\mathbb{Z})^\times\): 1, -1, 5, 13, -13, 17.

Answer: 1, 2, 6, 3, 6, 2. I won’t give explanations for all the answers here, just for 5. We have modulo 36, \( 5^1 = 5, 5^2 = 25, 5^3 = 17, 5^4 = 13, 5^5 = 29, 5^6 = 1 \). Therefore the least positive power of 5 which is the identity is 6, consequently the order of 5 is 6.

3. **1.1.25 on page 22** Prove that if \( G \) is a group and \( x^2 = 1 \) for all \( x \in G \), then \( G \) is abelian.

Let \( x, y \in G \). Then \( x^2 = y^2 = (xy)^2 = 1 \). Therefore

\[
x^2y^2 = 1 = (xy)^2 = xyxy.
\]

Multiplying by \( x^{-1} \) on the left and \( y^{-1} \) on the right, we obtain \( xy = yx \) as required.