Homework 9, Math 5126, due Monday, April 10, at the beginning of class. *Starred problems must be done without peer discussion.*

1.) (Page 668, no. 6) Suppose that $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence of $R$-modules. Prove that $M$ is a Noetherian $R$-module if and only if $M'$ and $M''$ are Noetherian $R$-modules.

2.) (Page 668, no. 7) Prove that submodules, quotient modules, and finite direct sums of Noetherian $R$-modules are again Noetherian $R$-modules.

*3.) (Page 668, no. 8) If $R$ is a Noetherian ring, prove that $M$ is a Noetherian $R$-module if and only if $M$ is a finitely generated $R$-module. (Thus any submodule of a finitely generated module over a Noetherian ring is also finitely generated).

4.) Let $A$ be a subring of the ring $B$ with unity $1$ such that $1 \in A$. Assume that the set $B \setminus A$ is closed under multiplication. Show that $A$ is integrally closed in $B$.

5.) (Page 703, no. 5) Let $R$ be an integral domain with field of fractions $F$. Show that $F$ is a finitely generated $R$-module if and only if $R = F$.

*6.) (Page 704, no. 11) Suppose $R$ is an integrally closed integral domain with field of fractions $k$ and $p(x) \in R[x]$ is a monic polynomial. Show that if $p(x) = a(x)b(x)$ with monic polynomials $a(x), b(x) \in k[x]$ then $a(x), b(x) \in R[x]$. (Hint: Explain why the roots of both $a(x)$ and $b(x)$ are integral over $R$. Then use this fact to show that the coefficients of $a(x)$ and $b(x)$ are both integral over $R$.)