Homework 6, Math 5125, due Monday, October 17, at the beginning of class.

Starred problems must be done without peer discussion.

1.) (Page 147, no. 15) Prove that a group of order 351 has a normal Sylow $p$-subgroup for some $p$ dividing its order.

2.) (Page 147, no. 16) Let $|G| = pqr$ where $p$, $q$, and $r$ are primes with $p < q < r$. Prove that $G$ has a normal Sylow subgroup for either $p$, $q$, or $r$.

*3.) (Page 147, no. 22) Prove that if $|G| = 132$ then $G$ is not simple.

4.) (Page 147, no. 24) Prove that if $G$ is a group of order 231 then $Z(G)$ contains a Sylow 11-subgroup of $G$ and a Sylow 7 subgroup is normal in $G$.

*5.) (Page 147, no. 26) Let $G$ be a group of order 105. Prove that if a Sylow 3 subgroup of $G$ is normal then $G$ is abelian.

6.) (Page 147, no. 32) Let $P$ be a Sylow $p$-subgroup of $H$ and let $H$ be a subgroup of $K$. If $P$ is normal in $H$ and $H$ is normal in $K$, prove that $P$ is normal in $K$. Deduce that if $P$ is a Sylow $p$ subgroup of $G$ and $H = N_G(P)$, then $N_G(H) = H$.

*7.) (Page 148, no. 50) Prove that if $U$ and $W$ are normal subsets of a Sylow $p$ subgroup $P$ of $G$ then $U$ is conjugate to $W$ in $G$ if and only if $U$ is conjugate to $W$ in $N_G(P)$. Deduce that two elements in the center of $P$ are conjugate in $G$ if and only if they are conjugate in $N_G(P)$. (A subset $U$ of $P$ is normal in $P$ if $N_P(U) = P$.)

8.) (Page 166, no. 7) Let $p$ be a prime and let $A = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_n \rangle$ be an abelian $p$ group where the order of $x_i$ equals $p^{\alpha_i}$ for some $\alpha_i \geq 1$ for all $i$. Define the $p^{th}$ power map

$$\varphi : A \to A \text{ by } \varphi : x \to x^p$$

(a) Prove that $\varphi$ is a homomorphism.

(b) Describe the image and kernel of $\varphi$ in terms of the given generators.

(c) Prove both $\ker \varphi$ and $A / \text{im } \varphi$ have rank $n$ and prove that these groups are both isomorphic to the elementary abelian group $E_{p^n}$ of order $p^n$. 