Homework 5, Math 5125, due Monday, September 26, at the beginning of class.  
Starred problems must be done without peer discussion.

1) (Page 122, no. 6) Let $r$ and $s$ be the usual generators for the dihedral group of order 8 and let $N = \langle r^2 \rangle$. List the left cosets of $N$ in $D_8$ as $1N$, $rN$, $sN$, and $srN$. Label these cosets with the integers 1, 2, 3, 4 respectively. Exhibit the image of each element of $D_8$ under the representation $\pi_N$ of $D_8$ into $S_4$ obtained from the action of $D_8$ by left multiplication on the set of 4 left cosets of $N$ in $D_8$. Deduce that this representation is not faithful and prove that $\pi_N(D_8)$ is isomorphic to the Klein 4-group.

*2) (Page 122, no. 7) Let $Q_8$ be the quaternion group of order 8.
(a) Prove that $Q_8$ is isomorphic to a subgroup of $S_8$.
(b) Prove that $Q_8$ is not isomorphic to a subgroup of $S_n$ for any $n \leq 7$. (Hint: If $Q_8$ acts on any set $A$ of order $\leq 7$ show that the stabilizer of any point $a \in A$ must contain the subgroup $< -1 >$.)

3) (Page 131, no. 19 and 20)
(a) Assume $H$ is a normal subgroup of $G$, $\mathcal{K}$ is a conjugacy class of $G$ contained in $H$ and $x \in \mathcal{K}$. Prove that $\mathcal{K}$ is a union of $k$ conjugacy classes of equal size in $H$, where $k = |G : H C_G(x)|$. Deduce that a conjugacy class in $S_n$ which consists of even permutations is either a single conjugacy class under the action of $A_n$ or is a union of two classes of the same size in $A_n$.
(b) Let $\sigma \in A_n$. Show that all elements in the conjugacy class of $\sigma$ in $S_n$ are conjugate in $A_n$ if and only if $\sigma$ commutes with an odd permutation.

4) (Page 132, no. 34 and Page 138, no. 11) Let $p$ be a prime and let $P$ be a subgroup of $S_p$ of order $p$.
(a) Prove $|N_{S_p}(P)| = p(p - 1)$.
(b) Prove $N_{S_p}(P)/C_{S_p}(P) \cong \text{Aut}(P)$.

*5) (Page 138, no. 12) Let $G$ be a group of order 3825. Prove that if $H$ is a normal subgroup of order 17 in $G$ then $H$ is a subgroup of $Z(G)$.