1. Given a ring $R$, define $R(z)$ to be the set of all formal sums $a + bz$, $a, b \in R$, where $z^2 = 0$, $az = za$ for all $a \in R$, and $a + bz = c + dz$ if and only if $a = c$ and $b = d$.
(a) If $A$ is an ideal of the ring $R$, show that $A(z) = \{c + dz | c, d \in A\}$ is an ideal of $R(z)$ and
\[ \frac{R(z)}{A(z)} \cong \frac{R}{A}(z). \]
(b) If $R$ is a division ring, show that $R(z)$ has exactly three ideals.
(c) Show that
\[ R(z) \cong \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in R \right\}. \]

2.) (Page 582, no. 8) Suppose $K$ is a Galois extension of $F$ of degree $p^n$ for some prime $p$ and some $n \geq 1$. Show there are Galois extensions of $F$ contained in $K$ of degrees $p$ and $p^{n-1}$.

*3.) (Page 596, no. 5) Let $p$ be a prime and let $F$ be a field. Let $K$ be a Galois extension of $F$ whose Galois group is a $p$-group (i.e. the degree $[K : F]$ is a power of $p$). Such an extension is called a $p$-extension (note that $p$-extensions are Galois by definition).
(a) Let $L$ be a $p$-extension of $K$. Prove that the Galois closure of $L$ over $F$ is a $p$-extension of $F$.
(b) Give an example to show that (a) need not hold if $[K : F]$ is a power of $p$ but $K/F$ is not Galois.

4.) (Page 618, no. 14) Prove the polynomial $x^4 + px^2 + q \in \mathbb{Q}[x]$ is irreducible for any distinct odd primes $p$ and $q$ and has Galois group the dihedral group of order 8.

*5.) (Page 618. no. 15) Prove the polynomial $x^4 + px + p \in \mathbb{Q}[x]$ is irreducible for every prime $p$ and for $p \neq 3, 5$ has Galois group $S_4$. Prove the Galois group for $p = 3$ is dihedral of order 8 and for $p = 5$ is cyclic of order 4.