Homework 3, Math 5125, due Monday, September 12, at the beginning of class.  
*Starred problems must be done without peer discussion.*

1) (a) Let $G$ be a group and let $N$ be a normal subgroup of $G$. Prove that $G/N$ is abelian if and only if $[G,G]$ is a subgroup of $N$.
(b) (part of Theorem 8 page 196) Let $G$ be group. Prove if $G^n = 1$ then $G$ is nilpotent.  
(Hint: Prove that if $G^n = 1$ then $G^{n-i} \subseteq Z_i(G)$ for $0 \leq i \leq n$.)

2) (a) Prove that $Z(D_{2n})$ is nontrivial if and only if $n$ is even.
(b) Use part (a) to prove: $D_{2n}$ is nilpotent if and only if $n$ is a power of 2.

*3) (Page 198, no. 7) Prove that subgroups and quotient groups of nilpotent groups are nilpotent (your proof should work for infinite groups). Given an explicit example of a group $G$ which possesses a normal subgroup $H$ such that both $H$ and $G/H$ are nilpotent but $G$ is not nilpotent.

*4) (a) Let $G$ be a nilpotent group. Prove: Every nonidentity normal subgroup of $G$ intersects nontrivially with $Z(G)$.
(b) Let $G$ be a finite group. Prove: $G$ is nilpotent if and only if $Z(G/K)$ is nontrivial for all proper normal subgroups $K$ of $G$.

5) a) Let $G$ be a finite group and let $M$ be a maximal subgroup of $G$. (That means that $M \neq G$ but there are no subgroups of $G$ properly between $M$ and $G$.) Prove that $Z(G) \subseteq M$ or $[G,G] \subseteq M$.
(b) Let $G$ be a nilpotent group and let $M$ be a maximal subgroup of $G$. Prove that $M$ is normal in $G$. 
