Homework 2. Math 5125, due Monday, September 5, at the beginning of class.  
*Starred problems must be done without peer discussion.*

1) Let \( n \geq 3. \)
   (i) Prove that the alternating group \( A_n \) is generated by the set of 3-cycles.
   (ii) Prove: If \( N \) is a proper normal subgroup of \( S_n \) and \( N \) contains a 3-cycle then \( N = A_n. \)

2.) (Page 101 no. 9) Let \( p \) be a prime and let \( G \) be a group of order \( p^a m \) where \( p \) does not divide \( m. \) Assume \( P \) is a subgroup of \( G \) of order \( p^a \) and \( N \) is a normal subgroup of \( G \) of order \( p^b n, \) where \( p \) does not divide \( n. \) Prove that \( |P \cap N| = p^b \) and \( |PN/N| = p^{a-b}. \)
   (Do not use Sylow’s theorem in your proof.)

3) (Page 106 no. 5) Prove that subgroups and quotient groups of a solvable group are solvable.

4) (Page 106 no. 8) Let \( G \) be a finite group. Prove that the following are equivalent.
   (i) \( G \) is solvable
   (ii) \( G \) has a chain of subgroups

\[
1 = H_0 \leq H_1 \leq H_2 \leq \cdots \leq H_s = G
\]

such that each \( H_i \) is normal in \( H_{i+1} \) and \( H_{i+1}/H_i \) is cyclic, \( 0 \leq i \leq s - 1 \)

(iii) all composition factors of \( G \) are of prime order

(iv) \( G \) has a chain of subgroups:

\[
1 = N_0 \leq N_1 \leq N_2 \leq \cdots N_t = G
\]

such that each \( N_i \) is a normal subgroup of \( G \) and \( N_{i+1}/N_i \) is abelian for \( 0 \leq i \leq t - 1. \)

5.) Let \( G \) be a group, \( B \) be a subgroup of \( G, \) and \( N \) be a normal subgroup of \( G. \)
   Prove: If both \( B \) and \( N \) are solvable then \( BN \) is solvable.