Homework 10, Math 5125, due Monday, November 14, at the beginning of class. *Starred problems must be done without peer discussion.*

1.) (Page 311, no. 8) Prove that $K_1 = \mathbb{F}_{11}[x]/(x^2 + 1)$ and $K_2 = \mathbb{F}_{11}[y]/(y^2 + 2y + 2)$ are fields with 121 elements. Prove that the map which sends $p(\bar{x})$ of $K_1$ to the element $p(\bar{y} + 1)$ of $K_2$ is well defined and gives a ring (hence field) isomorphism from $K_1$ to $K_2$.

2.) (Page 530, no. 7) Prove that $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Conclude that $[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}] = 4$. Find an irreducible polynomial satisfied by $\sqrt{2} + \sqrt{3}$.

3.) (Page 530, no. 8) Let $F$ be a field of characteristic not equal to 2. Let $D_1$ and $D_2$ be elements of $F$, neither of which is a square in $F$. Prove that $F(\sqrt{D_1}, \sqrt{D_2})$ is of degree 4 over $F$ if $D_1 D_2$ is not a square in $F$ and is of degree 2 over $F$ otherwise. (When $F(\sqrt{D_1}, \sqrt{D_2})$ is of degree 4 over $F$ the field is called a biquadratic extension of $F$.)

*4.) (Page 530, no. 9) Let $F$ be a field of characteristic not equal to 2. Let $a, b$ be elements of the field $F$ with $b$ not a square in $F$. Prove that a necessary and sufficient condition for $\sqrt{a} + \sqrt{b} = \sqrt{m} + \sqrt{n}$ for some $m$ and $n$ in $F$ is that $a^2 - b$ is a square in $F$. Use this to determine when the field $\mathbb{Q}(\sqrt{a} + \sqrt{b})(a, b \in \mathbb{Q})$ is biquadratic over $\mathbb{Q}$.

5.) (Page 530, no. 13) Suppose that $F = \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ where $\alpha_i^2 \in \mathbb{Q}$ for $i = 1, 2, \ldots, n$. Prove that the cube root of 2 is not in $F$.

* 6.) (Page 530, no. 14) Prove that if $[F(\alpha) : F]$ is odd then $F(\alpha) = F(\alpha^2)$.