For the $G$ we have constructed:

$$-\frac{\partial G_2}{\partial z} = -\frac{\partial}{\partial z} \left( -\int_0^z F_1(x,y,t)dt \right) = F_1(x,y,z)$$

$$\frac{\partial G_1}{\partial z} = \frac{\partial}{\partial z} \left( \int_0^z F_2(x,y,t)dt - \int_0^y F_3(x,t,0)dt \right) = F_2(x,y,z)$$

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = \frac{\partial}{\partial x} \left( -\int_0^z F_1(x,y,t)dt \right) - \frac{\partial}{\partial y} \left( \int_0^z F_2(x,y,t)dt - \int_0^y F_3(x,t,0)dt \right)$$

$$= -\int_0^z \left( \frac{\partial F_1}{\partial x}(x,y,t) + \frac{\partial F_2}{\partial y}(x,y,t) \right) dt + F_3(x,y,0)$$

Now use $\nabla \cdot E = 0$. We have $\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$, which implies that

$$-\frac{\partial F_1}{\partial x}(x,y,t) - \frac{\partial F_2}{\partial y}(x,y,t) = \frac{\partial F_3}{\partial z}(x,y,t)$$

Thus,

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = \int_0^z \frac{\partial}{\partial t} F_3(x,y,t) dt + F_3(x,y,0) = \int_0^z F_3(x,y,t) dt + F_3(x,y,0) = 0$$

$$\therefore \nabla \times G = E$$

Example: Let $E(x,y,z) = i \epsilon^y - \lambda \cos z - j - z \epsilon^k$

Verify that $\nabla \cdot E = 0$ and find a $G$ such that $E = \nabla \times G$.

a) $\nabla \cdot E = \frac{\partial}{\partial x}(\epsilon^y) + \frac{\partial}{\partial y}(-\lambda \cos z) + \frac{\partial}{\partial z}(-z \epsilon^k) = \epsilon^y + 0 - \epsilon^y = 0$

b) $G_1 = \int_0^z F_2(x,y,t) dt - \int_0^y F_3(x,t,0) dt = \int_0^z -\lambda \cos t dt - \int_0^y (-\epsilon^k) dt$

$$= -\lambda \sin t \bigg|_0^z + 0 = -\lambda \sin z$$

$G_2 = -\int_0^z F_1(x,y,t) dt = -\int_0^z \epsilon^y dt = -\epsilon^y$

$G_3 = 0$. Therefore: $G = -\lambda \sin z i - \epsilon^y j - z \epsilon^k$ j

Check: $\nabla \times G =$

$$\begin{vmatrix}
i & j & k \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
-\lambda \sin z & -\epsilon^y z & 0
\end{vmatrix} = \frac{\partial}{\partial x}(\epsilon^y) - \frac{\partial}{\partial y}(-\lambda \cos z) + \frac{\partial}{\partial z}(-z \epsilon^k) = \epsilon^y + 0 - \epsilon^y = E$$
Example (Electromagnetics):

Maxwell's Equations:

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \nabla \cdot \vec{D} = \rho \]
\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \nabla \cdot \vec{B} = 0 \]

where: \( \mathbf{B} = \mu \mathbf{H} \) (permeability \( \mu \), permittivity \( \varepsilon \) and conductivity \( \sigma \) are positive constants characterizing the material.)

\( \mathbf{E} \) is the electric field vector
\( \mathbf{B} \) is the magnetic field vector
\( \rho \) is volume charge density.

Some observations:

a) Apply Stokes' Theorem to surface \( S \) having simple closed boundary curve \( C \).

\[ \oint_{C} \mathbf{E} \cdot d\mathbf{r} = \iint_{S} \nabla \times \mathbf{E} \cdot d\mathbf{S} = \]
\[ = \iint_{S} -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \iint_{S} \mathbf{B} \cdot d\mathbf{S} \]

Therefore, an electromotive force around loop \( C \) is created by the (negative) time rate of change of magnetic flux through \( C \).

b) In the electrostatic case (no time variation), \( \nabla \times \mathbf{E} = \mathbf{0} \).

\[ \therefore \mathbf{E} = -\nabla \phi \quad \text{where } \phi \text{ is a scalar potential.} \]

Assuming \( \varepsilon \) constant,

\[ \nabla \cdot \mathbf{D} = \nabla \cdot (\varepsilon \mathbf{E}) = \varepsilon \nabla \cdot \mathbf{E} = \rho \]

\[ \therefore \varepsilon \nabla \cdot (-\nabla \phi) = \rho \Rightarrow \nabla \cdot \nabla \phi = \nabla^{2} \phi = -\frac{\rho}{\varepsilon} \quad \text{(Poisson's equation)} \]

where \( \nabla \cdot \nabla \phi = \nabla^{2} \phi = \phi_{xx} + \phi_{yy} + \phi_{zz} \).

If \( \rho = 0 \) we obtain Laplace's equation: \( \nabla^{2} \phi = 0 \).

c) Since \( \nabla \cdot \mathbf{B} = 0 \) (no magnetic charge) there exists a vector potential \( \mathbf{A} \) such that

\[ \mathbf{B} = \nabla \times \mathbf{A} \]
d) The displacement current term, $\frac{\partial D}{\partial t} = \varepsilon \frac{\partial E}{\partial t}$, in the $\nabla \times \mathbf{H} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}$ equation gives rise to electromagnetic waves. Consider a region of space in which $\sigma = 0$ (i.e., no conductivity) and $\rho = 0$ (i.e., no electric charge). Then:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \nabla \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} \quad \nabla \cdot \mathbf{H} = 0$$

$$\therefore \nabla \times (\nabla \times \mathbf{E}) = -\mu \nabla \times (\frac{\partial \mathbf{H}}{\partial t}) = -\mu \frac{\partial \mathbf{A}}{\partial t} \text{ (where we assume that the interchange of temporal and spatial partial derivatives is valid.)}$$

$$\therefore \nabla \times (\nabla \times \mathbf{E}) = -\mu \frac{\partial}{\partial t} (\varepsilon \frac{\partial \mathbf{E}}{\partial t}) = -\mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

A useful vector identity is: $\nabla \times \nabla \Delta = \nabla (\nabla \cdot \Delta) - \nabla^2 \Delta$

where $\nabla^2 \Delta$ is defined to be:

$$\nabla^2 (A_1 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k}) = (\nabla^2 A_1) \mathbf{i} + (\nabla^2 A_2) \mathbf{j} + (\nabla^2 A_3) \mathbf{k}$$

$$\therefore \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Since $\nabla \cdot \mathbf{E} = 0$, each component of $\mathbf{E}$ satisfies

$$\nabla^2 \mathbf{E}_j = \mu \varepsilon \frac{\partial^2 \mathbf{E}_j}{\partial t^2}, \quad j = 1, 2, 3$$

This is the wave equation, i.e., $\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$, with wave speed $c = \sqrt{\mu / \varepsilon}$.

In one spatial dimension $\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u(x,t)}{\partial t^2}$ has general solution

$$u(x,t) = f(x-ct) + g(x+ct) \text{ where } f, g \text{ are arbitrary (smooth) functions representing right and left travelling waves.}$$

One can show that the components of $\mathbf{H}$ satisfy the same wave equation, i.e., $\nabla^2 \mathbf{H}_j = \mu \varepsilon \frac{\partial^2 \mathbf{H}_j}{\partial t^2}, \quad j = 1, 2, 3$. 
Two important boundary value problems that arise in mathematical physics involve Laplace's (or Poisson's) equation and appropriate boundary conditions. (We'll consider Poisson's equation). They are:

**Dirichlet problem:** Given region $\Omega \subseteq \mathbb{R}^3$ with boundary surface $\partial \Omega$, solve for potential $\phi(x,y,z)$ in $\Omega$ if:

$$\nabla^2 \phi = \rho \text{ in } \Omega \quad \text{and} \quad \phi = f \text{ on } \partial \Omega$$

i.e. $\phi$ satisfies Poisson's equation in $\Omega$ and has prescribed values (given by $f$) on the boundary.

**Neumann problem:**

Solve for potential $\phi(x,y,z)$ in $\Omega$ if:

$$\nabla^2 \phi = \rho \text{ in } \Omega \quad \text{and} \quad \nabla \phi \cdot \mathbf{n} = \frac{\partial \phi}{\partial n} = f \text{ on } \partial \Omega$$

In this problem, the normal derivative (in the outward direction) is specified on $\partial \Omega$.

(In electrostatics, specifying $\nabla \phi \cdot \mathbf{n}$ on $\partial \Omega$ is tantamount to specifying the normal component of the electric field on $\partial \Omega$.)

The Divergence Theorem can be used to prove uniqueness results for these two problems. We first consider some integration formulas.

**Remark:** These problems arise in the study of steady state (i.e. time-independent) heat flow. In these problems, $\phi(x,y,z)$ is the steady state temperature within the body. The Dirichlet problem corresponds to specifying the temperature on the boundary. The Neumann problem specifies the thermal flux on the boundary. $\rho$ would correspond to the presence of heat sources or sinks within the body.
Some Integration Formulas:

Assume all (scalar or vector-valued) functions sufficiently smooth:

i) Since $\nabla \cdot (g \mathbf{f}) = \nabla g \cdot \mathbf{f} + g \nabla \mathbf{f}$, applying the Divergence Theorem:

$$\iiint_{V} \nabla \cdot (g \mathbf{f}) \, dV = \iiint_{V} (\nabla g \cdot \mathbf{f} + g \nabla \cdot \mathbf{f}) \, dV = \iint_{S} (g \mathbf{f}) \cdot \mathbf{n} \, dS = \iint_{S} g \mathbf{f} \cdot \mathbf{n} \, dS$$

ii) Special case:

Let $g = \phi$ and $\mathbf{f} = \nabla \phi$. Then, noting that $\nabla \cdot \mathbf{f} = \nabla \cdot \nabla \phi = \nabla^{2} \phi$, (i) becomes

$$\iiint_{V} \nabla \cdot (\phi \nabla \phi) \, dV = \iiint_{V} (\nabla \phi \cdot \nabla \phi + \phi \nabla^{2} \phi) \, dV = \iint_{S} \phi \nabla \phi \cdot \mathbf{n} \, dS - \iint_{S} \nabla \phi \cdot \mathbf{n} \, dS$$

(normal derivative).

iii) (Prob. 34.3, p. 288): Note that: $\nabla \cdot (g \mathbf{f}) = g \nabla \cdot \mathbf{f} + \nabla g \times \mathbf{f}$

Applying Stokes' Theorem:

$$\oint_{C} (g \mathbf{f}) \cdot d\mathbf{r} = \oint_{C^{+}} g \mathbf{f} \cdot d\mathbf{r} = \iint_{S} (\nabla \cdot (g \mathbf{f})) \cdot \mathbf{n} \, dS = \iint_{S} (g \nabla \cdot \mathbf{f} + \nabla g \times \mathbf{f}) \cdot \mathbf{n} \, dS$$

Suppose $\mathbf{f} = \nabla \phi$. Then $\nabla \cdot \mathbf{f} = \nabla \times \nabla \phi = 0$ and:

$$\oint_{C^{+}} (g \nabla \phi) \cdot d\mathbf{r} = \iint_{S} (\nabla \times \nabla \phi) \cdot \mathbf{n} \, dS$$

Example (Prob 34.4, p. 288): Suppose $\Omega \subseteq \mathbb{R}^{3}$ with smooth boundary.

Assume $u: \mathbb{R}^{3} \rightarrow \mathbb{R}$ is smooth and $\Delta u = \nabla^{2} u = 0$ in $\Omega$. Show that:

$$\int_{\partial \Omega} \frac{\partial (u^{2})}{\partial n} \, dS \geq 0$$

and if $u$ is not constant the inequality is strict, i.e. $\int_{\partial \Omega} \frac{\partial (u^{2})}{\partial n} \, dS > 0$.
Consider $\nabla^2(u^2) = \nabla \cdot (\nabla u^2)$. Applying differentiation formulas:

$\nabla (u^2) = u \nabla u + (\nabla u) u = 2u \nabla u$

$\therefore \nabla \cdot (\nabla u^2) = 2 \nabla \cdot (u \nabla u) = 2 [\nabla u \cdot \nabla u + u \nabla^2 u]$

Since $\nabla^2 u = 0$, we have:

$\nabla \cdot (\nabla u^2) = 2 \nabla u \cdot \nabla u = 2 |\nabla u|^2$.

Applying the Divergence Theorem:

$$\iiint_S \nabla \cdot (\nabla u^2) dV = 2 \iiint_S |\nabla u|^2 dV = \iint_S \nabla (u^2) \cdot \mathbf{n} dS = \iint_D \nabla (u^2) dS$$

Note that $1 \nabla u \cdot \mathbf{n} = 0$ and $|\nabla u| = 0$ in $S$ $\iff$ $u = \text{constant}$. Therefore,

$$2 \iiint_S |\nabla u|^2 dV = \iint_D \nabla (u^2) dS \geq 0 \quad \text{and equality if} \ u = \text{constant}. \quad \text{and equality if} \ u = \text{constant}.$$  

The ideas underlying this example can be used to establish uniqueness results for the Dirichlet & Neumann problems.

**Dirichlet problem:**

We establish uniqueness by assuming the problem has 2 solutions and showing that the 2 solutions must be the same.

Let: $\nabla^2 \phi = \rho$ in $V$, $\phi = f$ on $\partial V$

$\nabla^2 \phi_1 = \rho$ in $V$, $\phi_1 = f$ on $\partial V$

Let $w = \phi_1 - \phi_2$. Then: $\nabla^2 w = \nabla^2 (\phi_1 - \phi_2) = \nabla^2 \phi_1 - \nabla^2 \phi_2 = 0$ in $V$ and $w = f - f = 0$ on $\partial V$.

Summarizing: $\nabla^2 w = 0$ in $V$, $w = 0$ on $\partial V$.

Use: $\iiint_V \nabla \cdot (w \nabla w) dV = \iiint_V (|\nabla w|^2 + w \nabla^2 w) dV = \iiint_{\partial V} w \nabla w \cdot \mathbf{n} dS$

$$= \iint_{\partial V} w \frac{\partial w}{\partial n} dS$$
Since \( \nabla^2 W = 0 \) in \( V \) and \( W = 0 \) on \( \partial V \), the equality becomes:

\[
\iiint_V |\nabla W|^2 \, dV = 0
\]

Since \( |\nabla W| \geq 0 \) and is continuous in \( V \), we obtain \( \nabla W = 0 \).

\( \therefore W = \text{constant} \). However, \( W = 0 \) on \( \partial V \). \( \therefore W = 0 \) and \( \phi_1 = \phi_2 \).

Neumann problem:

\[
\nabla^2 \phi = \rho \text{ in } V
\]

\[
\nabla \phi \cdot \mathbf{n} = \frac{\partial \phi}{\partial n} = f \text{ on } \partial V
\]

Again, assume 2 solutions, \( \phi_1 \) and \( \phi_2 \), and let \( W = \phi_1 - \phi_2 \).

Then:

\[
\nabla^2 W = 0 \text{ in } V \quad \text{and} \quad \frac{\partial W}{\partial n} = 0 \text{ on } \partial V.
\]

As before:

\[
\iiint_V \nabla \cdot (w \nabla w) \, dV = \iiint_V \left( |\nabla w|^2 + w \nabla^2 w \right) \, dV = \iint_{\partial V} w \frac{\partial w}{\partial n} \, ds
\]

Since \( \nabla^2 W = 0 \) in \( V \) and \( \frac{\partial W}{\partial n} = 0 \) on \( \partial V \),

we have:

\[
\iiint_V |\nabla W|^2 \, dV = 0 \Rightarrow |\nabla W| = 0 \text{ or } \nabla W = 0.
\]

\( \therefore W = \phi_1 - \phi_2 = \text{const.} \Rightarrow \phi_1 = \phi_2 + \text{const} \).

In this problem, therefore, two solutions can differ by an additive constant.

Remark: In electrostatics, the Dirichlet problem solution corresponds to the potential being specified on the boundary while the Neumann problem corresponds to the normal component of the electric field. In the Neumann problem, the potential is determined only to within an additive constant.
Green's Function:

The Divergence Theorem is used to develop the ideas underlying Green's Functions and integral solutions of the Dirichlet problem.

For simplicity, consider Laplace's equation.

Problem: \( \bigtriangledown^2 \phi (\mathbf{r}) = 0, \quad \mathbf{r} \in V \)
\( \phi = f \) on \( \partial V \)

Such a function \( G \) is referred to as a Green's function for volume \( V \).

Let \( \phi, \psi \) be two sufficiently smooth functions.

Then:
\[
\iiint V \nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) \, dV = \iiint V \nabla \phi \cdot \nabla \psi + \phi \nabla^2 \psi - \nabla \psi \cdot \nabla \phi \, dV
\]
\[
-\psi \nabla^2 \phi \, dV = \iiint V [\phi \nabla^2 \psi - \psi \nabla^2 \phi] \, dV \quad \text{since} \quad \nabla \phi \cdot \nabla \psi = \nabla \psi \cdot \nabla \phi.
\]

Using the Divergence Theorem:
\[
\iiint V [\phi \nabla^2 \psi - \psi \nabla^2 \phi] \, dV = \iint_{\partial V} [\phi \nabla \psi \cdot \nabla \phi] \cdot \mathbf{n} \, dS
\]
\[
= \iint_{\partial V} [\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}] \, dS
\]

Consider the function:
\[
S(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|} = \frac{1}{4\pi \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}
\]

Let \( \mathbf{r} \) be an arbitrary but fixed point in \( V \) and view \( S(\mathbf{r}, \mathbf{r}') \)
as a function of $\mathbf{r}'$. One can show that:

$$\nabla'^2 s(\mathbf{r}, \mathbf{r}') = 0, \quad \mathbf{r}' \neq \mathbf{r}.$$ 

(Recall that if $\rho$ is the spherical radius of a coordinate system centered at $\mathbf{r}$, i.e., $\rho = |\mathbf{r}' - \mathbf{r}|$, then:

$$\nabla'^2 (\frac{1}{\rho^2} \phi) = \frac{1}{\rho^2} \frac{1}{\rho^2} \left\{ \frac{d}{d\rho} \left( \frac{\rho^2 \phi'}{\rho^2} \right) + \phi + \phi' \right\} = 0, \quad \rho \neq 0.$$ 

Apply the Divergence Theorem to the punctured volume $V_\varepsilon$ shown,

$V_\varepsilon$ is volume $V$ with a sphere of radius $\varepsilon$, centered at $\mathbf{r}$, removed.

$S_\varepsilon$ is the surface of the sphere of radius $\varepsilon$ centered at $\mathbf{r}$.

$$\iiint_{V_\varepsilon} \left[ \phi(\mathbf{r}') \nabla'^2 s(\mathbf{r}, \mathbf{r}') - s(\mathbf{r}, \mathbf{r}') \nabla'^2 \phi(\mathbf{r}') \right] dV' = 0 = \oint_{S_\varepsilon} \left[ \phi(\mathbf{r}') \frac{\partial}{\partial n'} s(\mathbf{r}, \mathbf{r}') - s(\mathbf{r}, \mathbf{r}') \frac{\partial}{\partial n'} \phi(\mathbf{r}') \right] dS' + \iiint_{S_\varepsilon} \left[ \phi(\mathbf{r}') \frac{\partial}{\partial n'} s(\mathbf{r}, \mathbf{r}') - s(\mathbf{r}, \mathbf{r}') \frac{\partial}{\partial n'} \phi(\mathbf{r}') \right] dS'$

Note: $dS_\varepsilon = \varepsilon^2 d\Omega$, where $d\Omega$ denotes solid angle measured relative to center at $\mathbf{r}$.

$$s(\mathbf{r}, \mathbf{r}') \bigg|_{\mathbf{r}' \in S_\varepsilon} = \frac{1}{4\pi \varepsilon}$$

$$\frac{\partial}{\partial n'} s(\mathbf{r}, \mathbf{r}') \bigg|_{\mathbf{r}' \in S_\varepsilon} = \frac{1}{4\pi \varepsilon^2} \quad (n' \text{ direction is toward the center of the } S_\varepsilon \text{ sphere}).$$

$$\phi(\mathbf{r}') = f(\mathbf{r}') \text{ when } \mathbf{r}' \in S.$$
Let \( \varepsilon \downarrow 0 \). Then:

\[
\lim_{\varepsilon \to 0} \iint_{S_\varepsilon} \left[ \phi \frac{\partial}{\partial n} s - s \frac{\partial \phi}{\partial n} \right] dS' = \lim_{\varepsilon \to 0} \iint_{\partial V} \left[ \phi'(\varepsilon) \left( \frac{1}{4\pi \varepsilon^2} - \frac{1}{4\pi} \frac{\partial \phi}{\partial n} \right) \right] \varepsilon^2 dS
\]

\[
\quad = \phi(\varepsilon) \quad \text{since} \quad \iint_{\partial V} dS = 4\pi \quad \text{and} \quad \phi, \frac{\partial \phi}{\partial n} \text{continuous at } \varepsilon.
\]

\[
\phi(\varepsilon) = -\iiint_{V} \left[ f(\varepsilon') \frac{\partial \varphi(\varepsilon',\varepsilon')}{\partial n'} - \varphi(\varepsilon',\varepsilon') \frac{\partial f(\varepsilon')}{\partial n'} \right] dS'
\]

This formula is not explicit since \( \frac{\partial \varphi}{\partial n} \) is not known on \( \partial V \).

Suppose, however, that we can find a function \( \psi(\varepsilon,\varepsilon') \) such that

1) \( \nabla^2 \psi(\varepsilon,\varepsilon') = 0 \) in \( V \)

2) \( \psi + \psi = 0 \) if \( \varepsilon' \in \partial V \)

Then if we form the Green's function \( G = \psi + \psi \), we have:

\[
\phi(\varepsilon) = -\iiint_{\partial V} \frac{\partial G(\varepsilon',\varepsilon)}{\partial n'} f(\varepsilon') dS'
\]

Remark: For the sphere, \( \psi \) and hence \( G \) can be constructed using the Method of Images.
**Implicit Function Theorem** (cf Notes, p183)

**Example:** As some introductory background, consider the equation \( f(x,y) = 0 \). When does this equation determine \( y \) as a function of \( x \)? If so, what is the domain? As a special case consider:

\[
f(x,y) = x^2 + y^2 - 1 = 0.
\]

The graph of this equation is the unit circle:

![Graph of the unit circle](image)

In calculus, we anticipated that \( y \) could be determined to be a (differentiable) function of \( x \) when the concept of implicit differentiation was introduced.

Recall that:

\[
2x + 2y \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y}
\]

Looking at the graph, we see that we can define \( y = y(x) \) for any \( x \)-interval not including \( x = \pm 1 \) as an interior point (dy undefined tangent).

![Graph showing two branches](image)

A function \( y(x) \) cannot be defined on this domain.

**Theorem:** Let \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \) have continuous partial derivatives. If \( f(x_0, y_0) = 0 \) and \( \frac{\partial f}{\partial y}(x_0, y_0) \neq 0 \), then there exists a neighborhood of \( x_0 \) on which a unique function \( y(x) \) is defined. Moreover, \( y \) is continuously differentiable and

\[
\frac{dy}{dx} = -\left. \frac{\partial f}{\partial y} \right|_{y=y(x)} \left. \frac{dx}{dy} \right|_{y=y(x)}^{-1} = -\frac{1}{2y(x)}
\]

**Example:** For the function \( y_1(x) \) shown above, with \( x_0 = \frac{x_1}{2} \)

\[
\frac{dy}{dx} \bigg|_{x=x_0} = -\frac{1}{2y_1(x_0)} \cdot 2 \cdot \frac{x_1}{2} = -1.
\]
Note: The linearization of \( f(x, y) \) about \( (x_0, y_0) \) is:

\[
f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0)
\]

Setting the linearization equal to zero, we see that we can solve for \( y \) as a function of \( x \) only if \( f_y(x_0, y_0) \neq 0 \), i.e.: \( y - y_0 = \frac{f_x(x_0, y_0)(x-x_0)}{f_y(x_0, y_0)} \)

Therefore, if the linearization permits us to solve for \( y \) as a function of \( x \) (i.e. the line tangent to the graph at \( (x_0, y_0) \) is not vertical) the full nonlinear problem has a solution (i.e. \( y \) as a function of \( x \)) locally (in some neighborhood) about \( x_0 \). This will be the general theme.

Example: \( f(x, y, z) = 0 \) typically defines a surface in \( \mathbb{R}^3 \) (e.g. \( x^2 + y^2 + z^2 - 1 = 0 \)). When does this equation define \( z \) as a function of \( (x, y) \) for example? What is the domain?

Let \( (x_0, y_0, z_0) \) be a point on the surface, i.e. \( f(x_0, y_0, z_0) = 0 \). The linearization of \( f \) about \( (x_0, y_0, z_0) \) is:

\[
f(x, y, z) \approx f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x-x_0) + f_y(x_0, y_0, z_0)(y-y_0) + f_z(x_0, y_0, z_0)(z-z_0)
\]

Setting the linearization equal to zero, we can solve for \( z \) as a function of \( (x, y) \) if \( f_z(x_0, y_0, z_0) \neq 0 \), i.e.

\[
z - z_0 = -\frac{f_x(x_0, y_0, z_0)}{f_z(x_0, y_0, z_0)} (x-x_0) - \frac{f_y(x_0, y_0, z_0)}{f_z(x_0, y_0, z_0)} (y-y_0)
\]

Geometric interpretation:

\[
\nabla f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)\hat{i} + f_y(x_0, y_0, z_0)\hat{j} + f_z(x_0, y_0, z_0)\hat{k}
\]

is a vector perpendicular to the tangent plane at \( (x_0, y_0, z_0) \). We can solve for \( z \) if the plane is not parallel to the \( z \)-axis.
Again, one can show that if the linearization has a solution, then the nonlinear problem has a solution locally, i.e., in some neighborhood of \((x_0, y_0, z_0)\).

**Example:** \(f(x, y, z) = x^2 + y^2 + z^2 - 1 = 0\). \(\frac{\partial f}{\partial z}(x_0, y_0, z_0) = 2z_0\). If \(z_0 \neq 0\), \(f(x, y, z) = 0\) has a "local solution." The "bad points" lie on the equator of this sphere of radius 1, where the tangent planes are parallel to the z-axis.

**General Case:** \(n\) equations, \(n+k\) unknowns.

\[
\begin{align*}
\mathbf{f}: & \quad \mathbb{R}^{n+k} \rightarrow \mathbb{R}^n, \quad \mathbf{u} \in \mathbb{R}^k, \quad \mathbf{v} \in \mathbb{R}^n, \quad \mathbf{f}(\mathbf{u}, \mathbf{v}) = \mathbf{0} \\
\begin{bmatrix}
    f_1(u_1, \ldots, u_k, v_1, \ldots, v_n) \\
    \vdots \\
    f_n(u_1, \ldots, u_k, v_1, \ldots, v_n)
\end{bmatrix} =
\begin{bmatrix}
    0 \\
    \vdots \\
    0
\end{bmatrix}
\end{align*}
\]

When can these equations implicitly define \(\mathbf{v}\) as a function of \(\mathbf{u}\)?

Let \((\mathbf{u}_0, \mathbf{v}_0)\) \(\in S\) and \(\mathbf{f}(\mathbf{u}_0, \mathbf{v}_0) = \mathbf{0}\).

**Linearization about \((\mathbf{u}_0, \mathbf{v}_0)\):**

\[
\mathbf{f}(\mathbf{u}, \mathbf{v}) \approx \mathbf{f}(\mathbf{u}_0, \mathbf{v}_0) + \begin{bmatrix}
    f_1(u, v) \\
    \vdots \\
    f_n(u, v)
\end{bmatrix} \mathbf{A} + \begin{bmatrix}
    f_1(u_0, v_0) \\
    \vdots \\
    f_n(u_0, v_0)
\end{bmatrix} \mathbf{B}
\]

\[
\begin{bmatrix}
    f_1(u, v) \\
    \vdots \\
    f_n(u, v)
\end{bmatrix} = \begin{bmatrix}
    f_1(u_1, \ldots, u_k, v_1, \ldots, v_n) \\
    \vdots \\
    f_n(u_1, \ldots, u_k, v_1, \ldots, v_n)
\end{bmatrix} \mathbf{A} + \begin{bmatrix}
    f_1(u_0, v_0) \\
    \vdots \\
    f_n(u_0, v_0)
\end{bmatrix} \mathbf{B}
\]

\[
\begin{bmatrix}
    f_1(u, v) \\
    \vdots \\
    f_n(u, v)
\end{bmatrix} = \mathbf{0}
\]

\[
\mathbf{A} (n \times k) \quad \mathbf{B} (n \times n)
\]

Setting the linearization equal to \(\mathbf{0}\),

\[
\mathbf{A}(\mathbf{u} - \mathbf{u}_0) + \mathbf{B}(\mathbf{v} - \mathbf{v}_0) = \mathbf{0}
\]

or \(\mathbf{v} - \mathbf{v}_0 = -\mathbf{B}^{-1}\mathbf{A}(\mathbf{u} - \mathbf{u}_0)\) if \(\det \mathbf{B} \neq 0\).