**Integral Theorems:**

**Green's Theorem:** Green's Theorem relates an area integral over a planar region to a line integral around its boundary.

**Orientation:**

- Positive orientation (area on left of walker)
- Negative orientation

**Claim:** Let \( P(x,y) \) be \( C^1 \) in \( D \) and let \( C \) denote its boundary. Then:

\[
\oint_C P(x,y) \, dx = - \iint_D \frac{\partial P(x,y)}{\partial y} \, dA
\]

**Proof:**

\[
\iint_D \frac{\partial P(x,y)}{\partial y} \, dA = \int_a^b \int_{\psi_1(x)}^{\psi_2(x)} \frac{\partial P(x,y)}{\partial y} \, dy \, dx = - \int_a^b \left[ P(x,\psi_2(x)) - P(x,\psi_1(x)) \right] \, dx
\]

By a similar argument:

\[
\iint_D \frac{\partial Q(x,y)}{\partial x} \, dA = \oint_C Q(x,y) \, dy
\]
For a simple region (one that is both $x$-simple and $y$-simple) or a union of such regions, e.g.:

Green's Theorem: Let $D \subseteq \mathbb{R}^2$ be a union of simple regions with boundary $C$. Let $P(x,y)$ & $Q(x,y)$ be $C^1$. Then:

$$\oint_C P\,dx + Q\,dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \,dA$$

where $C^+$ indicates boundary traversal in the positive (counterclockwise) sense.

Remark: If $E(x,y) = P(x,y)i + Q(x,y)j$, then $P\,dx + Q\,dy = E \cdot dr$ and:

$$\nabla \times E = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = i(\partial Q/\partial x - \partial P/\partial y) + j(\partial P/\partial z - \partial Q/\partial y) + k(\partial Q/\partial x - \partial P/\partial z)$$

and Green's Theorem can be expressed as:

$$\oint_{C^+} E \cdot dr = \iint_D \nabla \times E \cdot k \,dA$$

We will later see that Green's Theorem represents a special (planar) version of Stokes' Theorem.

**Example:** Verify Green's Theorem.

\[ P(x,y) = x + y \]
\[ Q(x,y) = y^2 \]
\[ \oint_{C^+} P\,dx + Q\,dy: \]

1. On 1: \( y = 0, \, dy = 0 \) and \( \int_{C^+} P\,dx + Q\,dy = \int_0^2 (x+0)\,dx = \left. x^2 \right|_0^2 = 2 \)
2. The line is: \( y = 4 - 2x \), \( \therefore \, dy = -2\,dx \)

\[ \int_{C^+} P\,dx + Q\,dy = \int_0^2 (x+4-2x)\,dx + (4-2x)^2(-2\,dx) \]
\[ = \int_0^2 (-8x^2 + 31x - 28)\,dx = -8\int_0^2 x^3 + 31x^2 - 28x \bigg|_0^2 = \frac{46}{3} \]
3. On 3: \( x = 0, \, dx = 0 \) and \( \int_{C^+} P\,dx + Q\,dy = \int_4^0 y^2\,dy = \left. \frac{y^3}{3} \right|_4^0 = -\frac{64}{3} \)

\[ \therefore \oint_{C^+} P\,dx + Q\,dy = 2 + \frac{46}{3} - \frac{64}{3} = -4 \]

\[ \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\,dA = \iint_D (0-1)\,dA = -\int_0^2 \int_0^{4-2x} dy\,dx = -\int_0^2 (4-2x)\,dx = -\left. (4x-x^2) \right|_0^2 = -4 \]

Example: Consider the annulus shown.

\[ \iint_{D_1} (Q_x - P_y)\,dA = \int_1^2 + \int_2^3 + \int_3^4 + \int_{C^+} P\,dx + Q\,dy \]
\[ \iint_{D_2} (Q_x - P_y)\,dA = \int_5^6 + \int_6^7 + \int_7^8 + \int_{C^+} P\,dx + Q\,dy \]

Since \( \int_2^3 + \int_{C^+} P\,dx + Q\,dy = 0 \) and \( \int_7^8 + \int_{C^+} P\,dx + Q\,dy = 0 \), we have:
\[
\iint_D (Q_x - P_y) \, dA = \oint_{C_1} P \, dx + Q \, dy + \oint_{C_2} P \, dx + Q \, dy
\]

**Example:** Verify Green's Theorem.

\[P(x,y) = x - y \quad Q(x,y) = y^2\]

a) \(Q_x - P_y = 0 - (-1) = 1\) and
\[
\iint_D (Q_x - P_y) \, dA = \pi (r_2^2 - r_1^2)
\]

b) Parametrize \(C_2\) as:

\[x = r_2 \cos t, \quad 0 \leq t \leq 2\pi\]
\[y = r_2 \sin t\]

\[
\oint_{C_2} P \, dx + Q \, dy = \int_0^{2\pi} \left[ (r_2 \cos t - r_2 \sin t)(-r_2 \sin t) + (r_2^2 \sin^2 t)(r_2 \cos t) \right] \, dt
\]

\[= \int_0^{2\pi} \left[ r_2^2 (-\cos t \sin t + \sin^2 t) + r_2^3 \sin^2 t \cos t \right] \, dt\]

\[= \left[ r_2^2 \left( -\frac{1}{2} \cos^2 t + t\sin 2t + \frac{r_2^3}{3} \sin^3 t \right) \right]_0^{2\pi} = \pi r_2^2\]

Parametrize \(C_1\) as:

\[x = r_1 \cos t, \quad 0 \leq t \leq 2\pi\]
\[y = -r_1 \sin t\]

\[
\oint_{C_1} P \, dx + Q \, dy = \int_0^{2\pi} \left[ (r_1 \cos t + r_1 \sin t)(-r_1 \sin t) + (r_1^2 \sin^2 t)(r_1 \cos t) \right] \, dt
\]

\[= -\pi r_1^2\]

\[
\therefore \oint_{C_1} P \, dx + Q \, dy + \oint_{C_2} P \, dx + Q \, dy = \pi (r_2^2 - r_1^2)
\]
**Remark:** In the form \( \oint_{C} E \cdot d\mathbf{r} = \iint_{D} \nabla \times E \cdot \mathbf{k} \ dA \), it is obvious that Green's Theorem holds for gradient fields. If \( E = \nabla \phi \), then \( \oint_{C} \nabla \phi \cdot d\mathbf{r} = 0 \) and \( \nabla \times \nabla \phi = 0 \).

**Examples:** Show that if \( C \) is a simple closed curve that bounds a region to which Green's Theorem applies, then the area of region \( D \) bounded by \( C \) is:

\[
A = \oint_{C} x \ dy - y \ dx
\]

\[
\therefore A = \frac{1}{2} \oint_{C} x \ dy - y \ dx
\]

**a)** Let \( P = 0, Q = x \). Then \( Q_{x} - P_{y} = 1 \) and Green's Theorem leads to

\[
\iint_{D} (Q_{x} - P_{y}) \ dA = \iint_{D} 1 \ dA = A = \oint_{C} (0 \ dx + x \ dy) = \oint_{C} x \ dy
\]

**b)** Let \( P = -y, Q = 0 \). Then \( Q_{x} - P_{y} = 1 \) and \( A = \oint_{C} (-y \ dx + 0 \ dy) = -\oint_{C} y \ dx \)

2. Use Green's Theorem (and the above result) to find the area under one arc of the cycloid \( x = \alpha(\theta - \sin \theta), y = \alpha(1 - \cos \theta), \alpha > 0, 0 \leq \theta \leq 2\pi \) bounded below by the \( x \)-axis.

![Cycloid Graph]

\[
A = \oint_{C} y \ dx = \oint_{C_{1}} y \ dx + \oint_{C_{2}} y \ dx
\]

**i)** Since \( y = 0 \) on \( C_{1} \), \( \oint_{C_{1}} y \ dx = 0 \)

**ii)** \( -\oint_{C_{2}} y \ dx = -\int_{0}^{2\pi} y \ dx \ d\theta = \int_{0}^{2\pi} \alpha(1 - \cos \theta) \cdot \alpha(1 - \cos \theta) \ d\theta = \alpha^{2} \int_{0}^{2\pi} (1 - \cos \theta)^2 \ d\theta \)

\[
= \alpha^{2} \left[ ( -2 \cos \theta + \frac{1}{2} + \frac{1}{2} \cos 2\theta) \right]_{0}^{2\pi} = \alpha^{2} \left( \frac{1}{2} \pi - 2 \sin \theta + \frac{1}{4} \sin 2\theta \right)_{0}^{2\pi} = 3\pi \alpha^{2}
\]

\[
\therefore \text{Area of } D = 3\pi \alpha^{2}
\]
Complex regions: Green's Theorem can be made to apply if we can subdivide the region as we did with the annulus.

Example:

Each slit is traversed in both directions, leading to a net contribution of zero.

The boundary is traversed so that region \( D \) always lies on the walker's left.

\[
\oint_{\partial D} (P \, dx - Q \, dy) = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]

Divergence Theorem in the Plane:

We will use Green's Theorem to derive a special 2-dimensional (planar) version of the Divergence Theorem. We'll later see how the standard 3-dimensional Divergence Theorem can be used to obtain this result.

Let \( D \) be a planar region to which Green's Theorem applies. Let \( C \) denote its boundary and let \( \vec{n} \) be a unit outward normal vector to \( C \). Let \( \vec{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j} \) be a \( C^4 \) vector field on \( D \). Then:

\[
\iint_D \nabla \cdot \vec{F} \, dA = \oint_C \vec{F} \cdot \vec{n} \, ds
\]

For the application of Green's Theorem, boundary \( C \) is traversed in CCW direction.
Proof:

Let \( C(t) = \langle x(t) \rangle + \gamma(t) \rangle \), \( a \leq t \leq b \), parametrize boundary \( C \) in the positive (counterclockwise) direction. Then:

\[ C(t) = x'(t) \langle i \rangle + y'(t) \langle j \rangle \] is tangent to \( C \) and:

\[ \mathbf{T}(t) = \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} (x'(t) \langle i \rangle + y'(t) \langle j \rangle) \] is a unit tangent vector.

\[ \mathbf{N} = \mathbf{T} \times \mathbf{K} = \frac{y'(t) \langle i \rangle - x'(t) \langle j \rangle}{\sqrt{(x'(t))^2 + (y'(t))^2}} \]

\[ ds = \sqrt{(x''(t))^2 + (y''(t))^2} \, dt \]

\[ \mathbf{E} \cdot \mathbf{N} ds = (y'(t) \langle i \rangle - x'(t) \langle j \rangle) \, dt \]

and

\[ \oint_C \mathbf{E} \cdot \mathbf{N} ds = \int_a^b \left[ P(x(t), y(t)) \langle i \rangle + Q(x(t), y(t)) \langle j \rangle \right] \cdot [y'(t) \langle i \rangle - x'(t) \langle j \rangle] \, dt \]

\[ = \int_a^b \left[ \frac{dP}{dt} - \frac{dQ}{dt} \right] dt = \int_C P \, dy - Q \, dx \]

Now apply Green's Theorem:

\[ \oint_C (-Q) \, dx + (P) \, dy = \iint_D \left[ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] \, dA = \iint_D \nabla \cdot \mathbf{F} \, dA \]

Remark: Our use of Green's Theorem to derive the result led us to use a counterclockwise traversal of boundary \( C \). The resulting integral \( \oint_C \mathbf{E} \cdot \mathbf{N} ds \), however, is insensitive to the manner in which the contributions from the differential arclength segments are summed.
Example: Verify the 2-dimensional divergence theorem for 

\[ E(x, y) = x^2y \mathbf{j} + (x + y^2) \mathbf{j} \]

and region \( D \) as shown.

\[ \nabla \cdot E = \frac{\partial}{\partial x} (x^2y) + \frac{\partial}{\partial y} (x + y^2) = 2xy + 2y \]

\[ \oiint_D \nabla \cdot E \, dA = \int_0^2 \int_0^1 2y(x+1) \, dy \, dx = \int_0^2 y^2(x+1) \bigg|_{y=0}^{y=1} \, dx = \int_0^2 (x+1) \, dx = \frac{1}{2} (x+1)^2 \bigg|_0^2 = \frac{1}{2} (9-1) = 4 \]

On \( \partial \): \( y = 0, \quad \mathbf{n} = -\mathbf{j}, \quad ds = dx \)

\[ \oint_{\partial} E \cdot \mathbf{n} \, ds = \int_0^1 (0 \mathbf{i} + x \mathbf{j}) \cdot (-\mathbf{j}) \, dx = -\frac{1}{2} x^2 \bigg|_0^1 = -2 \]

On \( \partial_2 \): \( x = 2, \quad \mathbf{n} = \mathbf{i}, \quad ds = dy \)

\[ \oint_{\partial_2} E \cdot \mathbf{n} \, ds = \int_0^1 (4y \mathbf{i} + (2 + y^2) \mathbf{j}) \cdot (\mathbf{i}) \, dy = \int_0^1 4y \, dy = 2y^2 \bigg|_0^1 = 2 \]

\( \partial_3 \): \( y = 1, \quad \mathbf{n} = \mathbf{j}, \quad ds = -dx \)

\[ \oint_{\partial_3} E \cdot \mathbf{n} \, ds = \int_0^1 (x^2 \mathbf{i} + (x+1) \mathbf{j}) \cdot (\mathbf{j}) \, (-dx) = \int_0^1 (x+1) \, dx = \frac{1}{2} (x+1)^2 \bigg|_0^2 = 4 \]

\( \partial_4 \): \( x = 0, \quad \mathbf{n} = -\mathbf{i}, \quad ds = -dy \)

\[ \oint_{\partial_4} E \cdot \mathbf{n} \, ds = \int_1^0 (0 \mathbf{i} + y^2 \mathbf{j}) \cdot (-\mathbf{i}) \, (-dy) = 0 \]

\[ \oint_D E \cdot \mathbf{n} \, ds = -2 + 2 + 4 + 0 = 4 \]

Example: Verify the 2-dimensional divergence theorem for 

\[ E(r, \theta) = r^3 \cos \theta \mathbf{\hat{r}} + \sin \theta \mathbf{\hat{\theta}} \]

and region \( D \) the annulus shown.

\[ \nabla \cdot E = \frac{1}{r} \frac{\partial}{\partial r} \left(r^3 \cos \theta \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \right) \]

\[ = \frac{1}{r} \left\{ 4r^3 \cos \theta + \cos \theta \right\} = (4r^2 + r^{-1}) \cos \theta \]

\[ dA = r \, dr \, d\theta \]

\[ \oiint_D (4r^2 + r^{-1}) \cos \theta \, dr \, d\theta = \int_0^{2\pi} \int_1^2 (4r^2 + r^{-1}) \cos \theta \, r \, dr \, d\theta \]

\[ = \int_0^{2\pi} \left( \frac{r^4}{4} + r \right) \bigg|_1^2 \cos \theta \, d\theta = 0 \]
On $\Omega: r = r_1, \mathbf{n} = -\mathbf{e}_r, ds = r_1 d\theta$

$$\int_{\Omega} \mathbf{E} \cdot d\mathbf{s} = \int_{0}^{2\pi} (r_1^3 \cos \theta \mathbf{e}_r + \sin \theta \mathbf{e}_\phi) \cdot (\mathbf{e}_r) r_1 d\theta =$$

$$= - \int_{0}^{2\pi} r_1^4 \cos \theta d\theta = 0$$

On $\Phi: r = r_2, \mathbf{n} = \mathbf{e}_r, ds = r_2 d\theta$

$$\int_{\Phi} \mathbf{E} \cdot d\mathbf{s} = \int_{0}^{2\pi} (r_2^3 \cos \theta \mathbf{e}_r + \sin \theta \mathbf{e}_\phi) \cdot (\mathbf{e}_r) r_2 d\theta =$$

$$= \int_{0}^{2\pi} r_2^4 \cos \theta d\theta = 0$$

$$\therefore \oint \mathbf{E} \cdot d\mathbf{s} = 0.$$

**Stokes' Theorem:**

Let $S \subset \mathbb{R}^3$ be an oriented surface with unit normal $\mathbf{n}$.

Assume that the boundary curve $C$ is oriented in the direction of $\mathbf{n}$ (Recall the right hand rule). Let $\mathbf{E}$ be a $C^4$ vector field. Then:

$$\iint_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = \oint_C \mathbf{E} \cdot d\mathbf{r}$$

**Example:**

Let $\mathbf{E}(\rho, \phi, \theta) = \rho^2 \mathbf{e}_\theta$ and let $S$ be the hemisphere shown.

\[ \nabla \times \mathbf{E} = \frac{1}{\rho^2 \sin \phi} \begin{vmatrix} \mathbf{e}_\rho & \mathbf{e}_\phi & \mathbf{e}_\theta \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ 0 & 0 & \rho^2 \sin \phi \end{vmatrix} = \frac{1}{\rho^2 \sin \phi} \left( \mathbf{e}_\rho (\rho \cos \phi) - \mathbf{e}_\phi (3 \rho^2 \sin \phi) + 0 \mathbf{e}_\theta \right) \]

$$= \mathbf{e}_\rho \rho \cot \phi - \mathbf{e}_\phi 3 \rho$$

On $S: \rho = \alpha; \mathbf{n} = \mathbf{e}_\rho$ and $dS = \alpha^2 \sin \phi d\phi d\theta$
\[ \iint_S \nabla \times \mathbf{E} \cdot \mathbf{n} \, dS = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left( \frac{1}{r} \mathbf{e}_\phi \right) \cdot \mathbf{e}_\phi \, a^2 \sin \phi \, d\phi \, d\theta = \]
\[ = a^3 \int_0^{2\pi} \cos \phi \, d\phi \, d\theta = a^3 \int_0^{2\pi} (\sin \phi)^{\frac{3}{2}} \, d\theta = 2\pi a^3 \]

Consider the boundary curve:
\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} a^2 \mathbf{e}_\phi \cdot (a \, d\theta \, \mathbf{e}_\theta) = \]
\[ = a^3 \int_0^{2\pi} d\theta = 2\pi a^3 \]

**Remark:** For a closed surface bounding a volume similar to the one shown,
\[ \iint_S \nabla \times \mathbf{E} \cdot \mathbf{n} \, dS = 0 \]

To see this, imagine the volume split as shown. Applying Stokes' Theorem to the "half-surfaces" \( S_1 \) and \( S_2 \), the integrals around boundary curves \( C_1 \) and \( C_2 \) are equal and opposite, cancelling each other.

\[ \iint_S \nabla \times \mathbf{E} \cdot \mathbf{n} \, dS = \]
\[ = \iint_{S_1} \nabla \times \mathbf{E} \cdot \mathbf{n} \, dS + \iint_{S_2} \nabla \times \mathbf{E} \cdot \mathbf{n} \, dS = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0 \]

We will not offer a proof of Stokes' Theorem for the general case. However, we will prove it in the case where surface \( S \) is the graph of a function \( f: D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3 \), i.e. \( z = f(x, y) \).