If: 
(i) \( \lambda_1 > 0, \lambda_2 > 0 \), \( f(x, y) - f(x_0, y_0) \geq 0 \) and \( (x_0, y_0) \) is a local minimum.
(ii) \( \lambda_1 < 0, \lambda_2 < 0 \), \( f(x, y) - f(x_0, y_0) \leq 0 \) and \( (x_0, y_0) \) is a local maximum.
(iii) \( \lambda_1 < 0, \lambda_2 > 0 \) or \( \lambda_1 > 0, \lambda_2 < 0 \), \( (x_0, y_0) \) is a saddle point.

Recall: 
\[
\lambda_{1,2} = \frac{a+c \pm \sqrt{(a+c)^2 - 4(ac-b^2)}}{2}
\]

(i) \( \lambda_1 > 0, \lambda_2 > 0 \) if \( a > 0 \) \& \( ac-b^2 > 0 \) (\( c > 0 \) also)
(ii) \( \lambda_1 < 0, \lambda_2 < 0 \) if \( a < 0 \) \& \( ac-b^2 > 0 \) (\( c < 0 \) also)
(iii) \( \lambda_1 > 0, \lambda_2 < 0 \) if \( ac-b^2 < 0 \)

Classification of Local Extrema:

(a) \( (x_0, y_0) \) is a local minimum of \( f(x, y) \) if:
   (i) \( f_x(x_0, y_0) = f_y(x_0, y_0) = 0 \)
   (ii) \( f_{xx}(x_0, y_0) > 0 \) and \( f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 > 0 \)

(b) \( (x_0, y_0) \) is a local maximum of \( f(x, y) \) if:
   (i) \( f_x(x_0, y_0) = f_y(x_0, y_0) = 0 \)
   (ii) \( f_{xx}(x_0, y_0) < 0 \) and \( f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 > 0 \)

(c) \( (x_0, y_0) \) is a saddle point of \( f(x, y) \) if:
   (i) \( f_x(x_0, y_0) = f_y(x_0, y_0) = 0 \)
   (ii) \( f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 < 0 \)

Remark: The classification does not cover all possibilities.

For example, \( f(x, y) = x^4 y^4 \) has a local minimum at \( (0, 0) \) but \( f_x, f_y, f_{xx}, f_{xy}, f_{yy} \) all vanish at \( (0, 0) \).
Example: \( f(x, y) = \frac{x^2}{2} + 2xy + y^2 + 3x + 2y \). Locate and classify the critical point(s) of \( f \).

\[ f_x(x, y) = x + 2y + 3, \quad f_y(x, y) = 2x + 2y + 2. \]

Require \( f_x(x_0, y_0) = 0 \) and \( f_y(x_0, y_0) = 0 \).

\[
\begin{bmatrix}
 1 & 2 \\
 2 & 2
\end{bmatrix}
\begin{bmatrix}
  x_0 \\
  y_0
\end{bmatrix}
= 

\begin{bmatrix}
  -3 \\
  -2
\end{bmatrix} \Rightarrow 

\begin{bmatrix}
  x_0 \\
  y_0
\end{bmatrix}
= 

\begin{bmatrix}
  -1 \\
  2
\end{bmatrix}

\begin{bmatrix}
  2 \\
  1
\end{bmatrix}

\begin{bmatrix}
  -3 \\
  -2
\end{bmatrix} = 

\begin{bmatrix}
  1 \\
  2
\end{bmatrix} \therefore \text{Critical pt.} \\
(x_0, y_0) = (1, -2),
\]

\[ f_{xx}(x, y) = 1, \quad f_{xy}(x, y) = 2, \quad f_{yy}(x, y) = 2 \]

\[ \therefore f_{xx}(1, -2) > 0, \quad f_{xx}(1, -2) f_{yy}(1, -2) - f_{xy}(1, -2)^2 = 2 - 4 < 0 \]

\[ \therefore (1, -2) \text{ is a saddle point.} \]

Generalization:

\( f: \mathbb{R}^n \to \mathbb{R} \). In particular, let \( n=3 \), \( f = f(x, y, z) \)

Taylor series:

\[
f(x, y, z) = f(x_0, y_0, z_0) + \left[ f_x(x_0, y_0, z_0) (x-x_0) + f_y(x_0, y_0, z_0) (y-y_0) + f_z(x_0, y_0, z_0) (z-z_0) \right] + 
\frac{1}{2} \left[ (x-x_0)^2, (y-y_0)^2, (z-z_0)^2 \right] 
\]

At critical point \((x_0, y_0, z_0)\):

\( \nabla f(x_0, y_0, z_0) = 0 \iff f_x(x_0, y_0, z_0) = f_y(x_0, y_0, z_0) = f_z(x_0, y_0, z_0) = 0 \)

(i) if the Hessian \( \frac{1}{2} (x-x_0)^T A (x-x_0) \) is positive definite (i.e. positive unless \( x-x_0 = 0 \)), \( x_0 \) is a local minimum. In this case \( A \) has 3 positive eigenvalues.

(ii) if the Hessian is negative definite (negative unless \( x=x_0 \)), \( x_0 \) is a local maximum. (\( A \) has 3 negative eigenvalues.)
Example: \( f: \mathbb{R}^3 \rightarrow \mathbb{R}, \quad f(x, y, z) = x^2 + y^2 + z^2 - 2x - 4y - 11 \)

Locate and classify (if possible) the critical points of \( f \).

\[ f_x(x, y, z) = 2x - 2, \quad f_y(x, y, z) = 2y - 4, \quad f_z(x, y, z) = 2z \]

Critical point: \((x_0, y_0, z_0) = (1, 2, 0)\)

At \((1, 2, 0)\):
\[
\begin{align*}
 f_{xx} &= 2, \quad f_{xy} = 0, \quad f_{xz} = 0 \\
 f_{yx} &= 0, \quad f_{yy} = 2, \quad f_{yz} = 0 \\
 f_{zx} &= 0, \quad f_{zy} = 0, \quad f_{zz} = 2
\end{align*}
\]

The Hessian is:
\[
\frac{1}{2} \begin{bmatrix}
(x-1), (y-2), z
\end{bmatrix} = \begin{bmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{bmatrix} \begin{bmatrix}
x-1 \\
y-2 \\
z
\end{bmatrix} = (x-1)^2 + (y-2)^2 + z^2
\]

Positive definite

\((1, 2, 0)\) is a local minimum.

Note: \( f(x, y, z) = (x-1)^2 + (y-2)^2 + z^2 - 16 \)

**Implicit Function Theorem:**

Consider the simple case \( y = f(x) \) \((f \in C^1)\) and assume \( \frac{dy}{dx} = f'(x) \neq 0 \) at \( x = x_0 \). (\( \therefore \) tangent line not horizontal.) Locally, one can define an inverse function \( x = f^{-1}(y) \) and:

\[
\frac{dx}{dy} = (f^{-1})'(y) \bigg|_{y = y_0} = \frac{1}{f'(x)} \bigg|_{x = x_0} = \frac{1}{f'(x)}
\]

Now consider \( F: \mathbb{R}^3 \rightarrow \mathbb{R}, \quad F \in C^1 \) \((\text{see p.33a})\)

Assume \( F(x_0, y_0, z_0) = 0 \) and \( \frac{\partial F}{\partial z}(x_0, y_0, z_0) \neq 0 \). Then there exists a disk \( U = \{(x, y) : \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \} \) and a neighborhood \( V = \{z : |z-z_0| < \varepsilon\} \) such that there is a unique function \( z = g(x, y) \) for \((x, y) \in U\), \( z \in V \) satisfying \( F(x, y, g(x, y)) = 0 \). \( z \) is continuously differentiable with partial derivatives determined as follows:
Given level curve \( F(x, y) = c \), when is it possible near \((x_0, y_0)\) on this curve to have it locally define \( y \) as a differentiable function of \( x \)?

Recall implicit differentiation. If \( y = y(x) \) and \( F(x, y(x)) = c \)
\[
\frac{d}{dx} F(x, y(x)) = \frac{\partial F}{\partial x}(x, y(x)) + \frac{\partial F}{\partial y}(x, y(x)) y'(x) = 0 \quad \text{(Chain Rule)}.
\]

\[
\therefore \quad y'(x) = -\frac{\frac{\partial F}{\partial x}(x, y(x))}{\frac{\partial F}{\partial y}(x, y(x))} \quad \text{assuming } \frac{\partial F}{\partial y}(x, y(x)) \neq 0.
\]

If \( \frac{\partial F}{\partial y}(x_0, y_0) \neq 0 \), one expects (with appropriate smoothness hypotheses) that the level curve \( F(x, y) = c \) implicitly defines \( y \) as a differentiable function of \( x \) in some neighborhood of \( x_0 \) with \( y'(x) \) given above.

**Geometric Interpretation.**

Since \( \nabla F(x, y) = \frac{\partial F}{\partial x}(x, y) \hat{i} + \frac{\partial F}{\partial y}(x, y) \hat{j} \), the requirement

\( \frac{\partial F}{\partial y}(x_0, y_0) \neq 0 \) is equivalent to demanding that the gradient not be parallel to the \( x \)-axis. Equivalently, the tangent line to the level curve at \((x_0, y_0)\) not be vertical.

**Example:** \( F(x, y) = \frac{x^2}{4} + y^2 = 1 \). Level curve is an ellipse.

\[ \nabla F(x, y) = \frac{x}{2} \hat{i} + 2y \hat{j} \]

\[ y' = -\frac{x}{4y} \quad \text{fails at } y = 0 \quad \text{where tangent line is vertical.} \]

At \((1, -\frac{1}{2})\):

\[ y' = -\frac{1}{4(-\frac{1}{2})} = \frac{1}{2\sqrt{3}}. \]
\[ \frac{\partial}{\partial x} F(x, y, z) = \frac{\partial}{\partial x} F(x, y, g(x, y)) = \frac{\partial F(x, y, g(x, y))}{\partial x} + \frac{\partial F(x, y, g(x, y))}{\partial z} \cdot \frac{\partial g}{\partial x} (x, y) \]

\[ \therefore \frac{\partial g}{\partial x} (x, y) = - \frac{\partial}{\partial x} F(x, y, g(x, y)) \]

\[ \frac{\partial}{\partial z} F(x, y, g(x, y)) \]

Similarly:

\[ \frac{\partial g}{\partial y} (x, y) = - \frac{\partial}{\partial y} F(x, y, g(x, y)) \]

\[ \frac{\partial}{\partial z} F(x, y, g(x, y)) \]

Remark: \( F(x_0, y_0, z_0) = 0 \Leftrightarrow (x_0, y_0, z_0) \) is a point on the level surface \( F(x, y, z) = 0 \).

\[ \frac{\partial F(x_0, y_0, z_0)}{\partial z} = \nabla F(x_0, y_0, z_0) \cdot k \neq 0 \] implies that the plane tangent to the level surface at \((x_0, y_0, z_0)\) is not parallel to the z-axis.

Example: \( F(x, y, z) = x^2 + y^2 + z^2 - c^2 \). \( F=0 \) is a sphere of radius \( c \) centered at the origin.

Let \( F(x_0, y_0, z_0) = 0 \). Then:

\[ \frac{\partial F(x_0, y_0, z_0)}{\partial z} = 2z_0 \neq 0 \] unless \((x_0, y_0, z_0)\) lies on the equator.
Chapter 4: Vector-Valued Functions

\( \mathbf{C} : \mathbb{R} \rightarrow \mathbb{R}^n, \ \mathbf{C}(t) = \begin{bmatrix} C_1(t) \\ \vdots \\ C_n(t) \end{bmatrix} \). For definiteness, we take \( n = 3 \),

\[ \mathbf{C}'(t) = \begin{bmatrix} C'_1(t) \\ C'_2(t) \\ C'_3(t) \end{bmatrix} \]

**Differentiation Formulas:**

(i) \( \frac{d}{dt}(\mathbf{b}(t)+\mathbf{c}(t)) = \mathbf{b}'(t)+\mathbf{c}'(t) \)

(ii) \( \frac{d}{dt}(p(t)\mathbf{c}(t)) = p'(t)\mathbf{c}(t)+p(t)\mathbf{c}'(t) \)

(iii) \( \frac{d}{dt}(\mathbf{b}(t) \cdot \mathbf{c}(t)) = \mathbf{b}'(t) \cdot \mathbf{c}(t)+\mathbf{b}(t) \cdot \mathbf{c}'(t) \)

(iv) \( \frac{d}{dt}(\mathbf{b}(t) \times \mathbf{c}(t)) = \mathbf{b}'(t) \times \mathbf{c}(t)+\mathbf{b}(t) \times \mathbf{c}'(t) \)

(v) \( \frac{d}{dt}(\mathbf{C}(q(t))) = \mathbf{C}'(q(t))q'(t) \)

**Proof:**

(iii) \( \frac{d}{dt}(\mathbf{b} \cdot \mathbf{c}) = \frac{d}{dt}(b_1c_1+b_2c_2+b_3c_3) = b'_1c_1+b'_2c_2+b'_3c_3 = (b_1c_1+b'_2c_2+b'_3c_3) + (b_1c'_1+b_2c'_2+b_3c'_3) = \mathbf{b}' \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}' \)

(iv) \( \frac{d}{dt}(\mathbf{b} \times \mathbf{c}) = \frac{d}{dt} \left| \begin{array}{ccc} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right| = \frac{d}{dt} \left( i(b_2c_3-b_3c_2)-j(b_3c_1-b_1c_3)+k(b_1c_2-b_2c_1) \right) = i(b_2c_3+b_2c'_3-b'_3c_2) - j(b_3c_1+b_3c'_3-b'_3c_1) + k(b_1c_2+b_1c'_2 - b'_2c_1) = \left[ i(b_2c_3-b_3c_2) - j(b_3c_1-b_1c_3) + k(b_1c_2-b_2c_1) \right] + \left[ i(b_2c'_3-b_3c'_2) - j(b_3c'_1-b_1c'_3) + k(b_1c'_2-b_2c'_1) \right] = \mathbf{b}'(t) \times \mathbf{c}(t) + \mathbf{b}(t) \times \mathbf{c}'(t) \)
Example: (Motion of a charged particle in a constant magnetic field $B$.)

Let $\mathbf{v}(t)$ denote the velocity vector. Show that kinetic energy $\frac{1}{2}m |\mathbf{v}(t)|^2$ is constant.

Newton's Law: $m \frac{d}{dt} \mathbf{v} = q (\mathbf{v} \times \mathbf{B})$ (Lorentz force) where

$m =$ mass
$q =$ charge
$\mathbf{B} =$ constant magnetic field.

$$\frac{d}{dt} \left( \frac{1}{2} m |\mathbf{v}|^2 \right) = \frac{1}{2} m \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v}) = \frac{1}{2} m (\mathbf{v}' \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}') = m (\mathbf{v}' \cdot \mathbf{v}) = q (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = 0$$

Arc length:

Let $\mathbf{r}(t)$, $a \leq t \leq b$, be the parametric description of a path in space. What is its length?

Polygonal approximation:

$$L(\mathbf{r}) \approx \sum_{i=1}^{n} |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})| \quad \text{where} \quad t_0 = a, t_n = b$$

$$= \sum_{i=1}^{n} \frac{|\Delta \mathbf{r}_i|}{\Delta t_i} \Delta t_i$$

As the partition norm goes to 0, we obtain:

$$L(\mathbf{r}) = \int_{a}^{b} |\mathbf{r}'(t)| \, dt$$

Remark: If $t$ is time, we get length equal to the integral of speed over time.

$$L(\mathbf{r}) = \int_{a}^{b} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt$$

Example: Compute the length of the curve:

$$\mathbf{r}(t) = \left( \ln(t^2), \sqrt{t}, \frac{3}{2} t^2 \right), \quad 1 \leq t \leq 2$$
\[ t'(t) = \left( \frac{1}{t^3}, \frac{1}{3t}, 3t \right), \quad |t'(t)| = \sqrt{\left( \frac{1}{t^2} \right)^2 + (3t)^2} = \sqrt{\frac{1}{t} + 3 + 9t^2} \]

\[ L(t) = \int_1^2 \left( \frac{1}{2t} + 3t \right) dt = \frac{1}{2} \ln(2) + 6 \left( \frac{5}{2} \right) = \frac{1}{2} \ln(2) + \frac{9}{2} \]

**Vector Fields:**

Each point in the domain is assigned a vector (force field, particle velocity in fluid flow).

**Gradient vector field:** Vector field (electrostatic, gravitational, etc.) is the gradient of a scalar potential.

**Example (gravitational force field)**

\[ \text{Potential} \quad V(x, y, z) = -\frac{mMG}{r} \quad \text{Force field} \quad \text{(created by } M \text{ acting on } m) : \]

\[ F = -\nabla V = -\frac{mMG}{r^3} \quad \text{where } \hat{r} \text{ is a unit vector in the radial direction.} \]

**Divergence:** Let \( F = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k} \). The divergence of \( F \) is the scalar function:

\[ \text{div } F = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \]

**Remark:** If we introduce the del operator:

\[ \nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \]

then \( \nabla \cdot F \) can be formally obtained as:

\[ (\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}) \cdot (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}) \]

**Interpretation (heuristic):**

Consider the volume element shown.
The net "flow" of \( \mathbf{F} \) out of the element is:

\[
(\mathbf{F}_1(x+\Delta x, y, z) - \mathbf{F}_1(x, y, z)) \Delta y \Delta z + (\mathbf{F}_2(x, y+\Delta y, z) - \mathbf{F}_2(x, y, z)) \Delta x \Delta z + \\
(\mathbf{F}_3(x, y, z+\Delta z) - \mathbf{F}_3(x, y, z)) \Delta x \Delta y = \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) \Delta V
\]

As the volume element shrinks, the net "flow" per unit volume approaches \( \nabla \cdot \mathbf{F} \).

In applications, regions where sources/sinks of \( \mathbf{F} \) are absent, are characterized by \( \nabla \cdot \mathbf{F} = 0 \).

(For example, in charge-free regions, the electrostatic field \( \mathbf{E} \) satisfies \( \nabla \cdot \mathbf{E} = 0 \). Similarly, the velocity field of an incompressible fluid satisfies \( \nabla \cdot \mathbf{v} = 0 \).)

The integral Divergence Theorem will provide a precise macroscopic interpretation.

**Example (p. 258#4)** \( \mathbf{V}(x, y, z) = x^2 \mathbf{i} + (x+y)^2 \mathbf{j} + (x+y+z)^2 \mathbf{k} \)

\[
\nabla \cdot \mathbf{V} = \frac{\partial}{\partial x} (x^2) + \frac{\partial}{\partial y} ((x+y)^2) + \frac{\partial}{\partial z} ((x+y+z)^2) \\
= 2x + 2(x+y) + 2(x+y+z) = 2x + 4y + 2z
\]
Curl of a Vector Field:

\[ \mathbf{E} = E_x(x,y,z) \mathbf{i} + E_y(x,y,z) \mathbf{j} + E_z(x,y,z) \mathbf{k} \]

Del operator:

\[ \nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \]

\[ \text{curl } \mathbf{E} = \nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \mathbf{i} - \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) \mathbf{j} + \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \mathbf{k} \]

(vector)

Examples:

(i) \[ \mathbf{E}(x,y,z) = xz \mathbf{i} + (z-y^2) \mathbf{j} + 3x^2 \mathbf{k} \]

\[ \nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & z-y^2 & 3x^2 \end{vmatrix} = \begin{vmatrix} x & -y & 3x^2 \\ z & 0 & -y \\ 1 & 2y & 0 \end{vmatrix} = (3x^2-1) \mathbf{i} - (6xy-x) \mathbf{j} + k(0-0) \]

(ii) \[ \mathbf{E} = x^2 \mathbf{i} + y^2 \mathbf{j} = \mathbf{E}(x,y,z) + 0 \mathbf{k} \]

\[ \nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0 \]

Laplacian:

\[ \nabla^2 f(x,y,z) = \Delta f(x,y,z) = \nabla \cdot (\nabla f(x,y,z)) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \]

Some Vector Identities: (see list, p. 255)

1. \[ \nabla \cdot (\nabla \mathbf{E}) = 0 \]
2. \[ \nabla \times (\nabla \mathbf{f}) = 0 \]
3. \[ \nabla \cdot (\mathbf{f} \mathbf{E}) = \nabla \mathbf{f} \cdot \mathbf{E} + \mathbf{f} \nabla \cdot \mathbf{E} \]
4. \[ \nabla \times (\mathbf{f} \mathbf{E}) = \nabla \mathbf{f} \times \mathbf{E} + \mathbf{f} \nabla \times \mathbf{E} \]
Proof of (i):

\[ F = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}, \]

\[ \nabla \times F = \frac{1}{r} \left( \frac{\partial}{\partial r} \left( F_2 r \hat{\phi} - F_3 r \hat{\theta} \right) - \frac{\partial}{\partial \theta} \left( F_2 r \hat{\theta} - F_3 r \hat{\phi} \right) + \frac{\partial}{\partial \phi} \left( F_3 r \hat{\theta} - F_1 r \hat{\phi} \right) \right) \]

\[ \nabla \cdot (\nabla \times F) = \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} (F_2 - F_3) \right) - \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} (F_2 - F_3) \right) + \frac{\partial}{\partial \phi} \left( \frac{\partial}{\partial \phi} (F_2 - F_3) \right) \]

\[ = \frac{\partial^2}{\partial r^2} (F_2 - F_3) + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} (F_2 - F_3) \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} (F_2 - F_3) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \phi} \left( \frac{\partial}{\partial \phi} (F_2 - F_3) \right) = 0, \]

using the equality of mixed second order partial derivatives.

Cylindrical and Spherical Coordinates:

Unit vectors:

Cylindrical coordinates:

\[ \hat{e}_r, \hat{e}_\theta, \hat{e}_z = \hat{k} \]

\[ \hat{e}_r = \cos \theta \hat{i} + \sin \theta \hat{j} \]

\[ \hat{e}_\theta = -\sin \theta \hat{i} + \cos \theta \hat{j} \]

\[ \hat{e}_z = \hat{k} \]

\[ \hat{e}_r, \hat{e}_\theta, \hat{e}_z \] form a set of three mutually perpendicular unit vectors. In contrast to \( \hat{i}, \hat{j}, \hat{k} \), as a point moves, the orientation of vectors \( \hat{e}_r, \hat{e}_\theta \) characterizing the point changes.

In particular, if the polar coordinate angle varies in time as \( \theta(t) \):

\[ \frac{d}{dt} \hat{e}_r = -\sin \theta \dot{\theta} \hat{i} + \cos \theta \dot{\theta} \hat{j} = \dot{\theta} \hat{e}_\theta \]

\[ \frac{d}{dt} \hat{e}_\theta = -\cos \theta \dot{\theta} \hat{i} - \sin \theta \dot{\theta} \hat{j} = -\dot{\theta} \hat{e}_r \]

\[ \frac{d}{dt} \hat{e}_z = \frac{d}{dt} (\hat{k}) = 0 \]

Note:

\[ \hat{e}_r \times \hat{e}_\theta = \hat{e}_z, \quad \hat{e}_\theta \times \hat{e}_z = \hat{e}_r, \quad \hat{e}_z \times \hat{e}_r = \hat{e}_\theta \]