1. (2009 Putnam exam, problem A1). Let \( f \) be a real-valued function on the plane such that for every square \( ABCD \) in the plane, \( f(A) + f(B) + f(C) + f(D) = 0 \). Does it follow that \( f(P) = 0 \) for all points \( P \) in the plane?

Yes, it does follow. Let \( P \) be any point in the plane. Let \( ABCD \) be any square with center \( P \). Let \( E, F, G, H \) be the midpoints of the segments \( AB, BC, CD, DA \), respectively. The function \( f \) must satisfy the equations

\[
\begin{align*}
0 &= f(A) + f(B) + f(C) + f(D) \\
0 &= f(E) + f(F) + f(G) + f(H) \\
0 &= f(A) + f(E) + f(P) + f(H) \\
0 &= f(B) + f(F) + f(P) + f(E) \\
0 &= f(C) + f(G) + f(P) + f(F) \\
0 &= f(D) + f(H) + f(P) + f(G).
\end{align*}
\]

If we add the last four equations, then subtract the first equation and twice the second equation, we obtain \( 0 = 4f(P) \), whence \( f(P) = 0 \).

2. Prove that for any integer \( n \geq 1 \), \( 2^{2n} - 1 \) is divisible by 3.

Base case \( n = 1 : 2^{2-1} - 1 = 4 - 1 = 3 \) is divisible by 3. Inductive step: Assume that \( n \geq 1 \) and \( 2^{2n} - 1 \) is divisible by 3, so \( 2^{2n} - 1 = 3k \). Then

\[
2^{2(n+1)} - 1 = 2^{2n+2} - 1 = 4 \cdot 2^{2n} - 1 = 3 \cdot 2^{2n} + (2^{2n} - 1) = 3 \cdot 2^{2n} + 3k,
\]

and this is a multiple of 3.

Other solutions: Since the polynomial \( x^{2n} - 1 \) has root \( x = -1 \) (among others), we can factor \( x^{2n} - 1 = (x + 1)(x^{2n-1} - x^{2n-2} + \ldots - 1) \). Put \( x = 2 \). Some of you did it by factoring \( 2^{2n} - 1 = (2^n - 1)(2^n + 1) \) and considering this product mod 3. One of the consecutive numbers \( 2^n - 1, 2^n, \) and \( 2^n + 1 \) must be divisible by 3. Certainly \( 2^n \) is not, so one of the other two factors must be divisible by 3.

3. (From the 2008 Putnam exam, problem B1). What is the maximum number of rational points that can lie on a circle in \( \mathbb{R}^2 \) whose center is not a rational point? (A rational point is a point both of whose coordinates are rational numbers.)

Answer: 2

There are at most two such points. For example, the points \((0,0)\) and \((1,0)\) lie on a circle with center \((1/2, x)\) for any real number \(x\), not necessarily rational.

On the other hand, suppose \( P = (a,b), Q = (c,d), R = (e,f) \) are three rational points that lie on a circle. The midpoint \( M \) of the side \( PQ \) is \( ((a+c)/2, (b+d)/2) \), which is again rational. Moreover, the slope of the line \( PQ \) is \((d-b)/(c-a)\), so the slope of the line through \( M \) perpendicular to \( PQ \) is \((a-c)/(b-d)\), which is rational or infinite.

Similarly, if \( N \) is the midpoint of \( QR \), then \( N \) is a rational point and the line through \( N \) perpendicular to \( QR \) has rational slope. The center of the circle lies on both of these lines, so its coordinates \((g,h)\) satisfy two linear equations with rational coefficients, say \( Ag + Bh = C \) and \( Dg + Eh = F \). Moreover, these equations have a unique solution. That solution must then be

\[
\begin{align*}
g &= (CE - BD)/(AE - BD) \\
h &= (AF - BC)/(AE - BD)
\end{align*}
\]

(by elementary algebra, or Cramer’s rule), so the center of the circle is rational.
For more details go to http://amc.maa.org/a-activities/a7-problems/putnamindex.shtml and download the solution to problem B1-2008.

4. Show that the sum
   \[ \sqrt{1001^2 + 1} + \sqrt{1002^2 + 1} + \cdots + \sqrt{2000^2 + 1} \]
   is not an integer.

   Use the estimate \( \sqrt{n^2 + 1} \leq n + \frac{1}{n} \) (or \( \sqrt{n^2 + 1} \leq n + \frac{1}{2n} \)) and note that \( \frac{1}{1001} + \cdots + \frac{1}{2000} < \frac{1000}{1001} < 1 \).