1. Prove that $n! > 2^n$ for all $n \geq 4$.

Solution: Use induction on $n$. For $n = 4$ we have $4! = 24 > 16 = 2^4$. This proves the base case.

Suppose that $k! > 2^k$ for any given $k \geq 4$. Then $(k+1)! = (k+1)k! > (k+1)2^k > 2 \cdot 2^k = 2^{k+1}$.

2. Let $r$ be a number such that $r + 1/r$ is an integer. Prove that for every positive integer $n$, $r^n + 1/r^n$ is an integer.

Solution: For $n = 1$, $r + 1/r \in \mathbb{Z}$ by assumption and for $n = 2$ it follows from $r^2 + 1/r^2 = (r + 1/r)(r + 1/r) - 2$.

Suppose that for a given $n$ it is true that $r^k + 1/r^k \in \mathbb{Z}$ for all $2 \leq k \leq n$. We have already verified this for $n = 2$. Then from $r^n + 1/r^n = (r^{n-1} + 1/r^{n-1})(r + 1/r) - (r^{n-2} + 1/r^{n-2})$ we see that the right-hand side is an integer.

3. Prove that $n! < \left(\frac{n+1}{2}\right)^n$, for $n = 2, 3...$ Use the Arithmetic Mean-Geometric Mean inequality:

$$\sqrt[2]{a_1 \cdot a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad a_k \geq 0.$$ 

Solution: $n! \leq \left(\frac{1 + 2 + \cdots + n}{n}\right)^n = \left(\frac{n(n+1)}{2n}\right)^n = \left(\frac{n+1}{2}\right)^n$.

The fact that $1 + 2 + \cdots + n = n(n+1)/2$ was used.

4. Evaluate (Putnam 2003)

$$I = \int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}} \quad (1)$$

Hint: What is $\tan(\pi/2 - x)$ equal to?

Solution: $\tan(\pi/2 - x) = \cot(x) = 1/\tan(x)$. So (make the substitution $x = \pi/2 - z$, $dx = -dz$)

$$I = \int_0^{\pi/2} \frac{dz}{1 + [\tan(\pi/2 - z)]^{\sqrt{2}}} = \int_0^{\pi/2} \frac{(\tan z)^{\sqrt{2}}dz}{1 + (\tan z)^{\sqrt{2}}} \quad (2)$$

Adding (1) and (2) gives

$$2I = \int_0^{\pi/2} \frac{1 + (\tan z)^{\sqrt{2}}}{1 + (\tan z)^{\sqrt{2}}} dz = \int_0^{\pi/2} dz = \frac{\pi}{2}.$$ 

Hence $I = \frac{\pi}{4}$.

5. Evaluate

$$I = \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(3+x)}} dx$$

Solution: Make the substitution $x = 3 + u$ and then $v = -u$ and proceed as in problem #4. The result is $I = 1$. 