Combinatorial Proofs Practice

Remember that a combinatorial proof has three parts:

1. State the set being counted.

2. Count the set one way (corresponding to the LHS of the equation)

3. Count the set another way (corresponding to the RHS of the equation)

All three steps should be clear in your proof. It is sometimes helpful to frame your sets in terms of committees with particular properties. In general multiplication in an equation tells you to choose one thing, and then something else as you count. Addition implies the set can be considered in disjoint cases.

1. Prove that

\[ \sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}. \]

\( \text{(Hint: Recall that } \binom{n}{k} = \binom{n}{n-k} \text{.)} \)

2. Prove that

\[ \binom{n}{k} = \binom{n-2}{k-2} + 2 \binom{n-2}{k-1} + \binom{n-2}{k}. \]

3. Prove that

\[ \binom{n}{r} \binom{r}{k} = \binom{n}{k} \binom{n-k}{r-k} \]

whenever \( k \leq r \leq n. \)

4. Prove that

\[ n^3 = 1 + 3(n-1) + 3(n-1)^2 + (n-1)^3. \]