Solutions 2.10

1.a) Integration yields
\[ y = t^2 - t + C, \]
and the initial condition yields \( C = 0. \)
b) \( y_{k+1} = y_k + h(2t_k - 1). \)
c) \( y_0 = 0, \ y_1 = 0 + 0.1 \times (2 \times 1 - 1) = 0.1, \ y_2 = 0.2 + 0.1 \times (2 \times 1.1 - 1) = 0.22, \ y_3 = 0.32 + 0.1 \times (2 \times 1.2 - 1) = 0.36. \)
d) \( y(1.1) = 0.11, \ y(1.2) = 0.24, \ y(1.3) = 0.39. \)
2.a) The exact solution is \( y = \exp(-t). \)
b) \( y_{k+1} = y_k - hy_k = (1 - h)y_k. \)
c) \( y_0 = 1, \ y_1 = 0.9, \ y_2 = 0.81, \ y_3 = 0.729. \)
d) \( y(0.1) = 0.9048, \ y(0.2) = 0.8187, \ y(0.3) = 0.7408. \)
3.a) The exact solution is \( y(t) = \exp(-t^2/2). \)
b) \( y_{k+1} = y_k - ht_k y_k. \)
c) \( y_0 = 1, \ y_1 = 1, \ y_2 = 0.99, \ y_3 = 0.9702. \)
d) \( y(0.1) = 0.9950, \ y(0.2) = 0.9802, \ y(0.3) = 0.9560. \)
Euler: $x' = \sin(2 \times 3.141592653 \times t) + t \times x$, $h = [0.01]$
Solutions 3.1

3. The singular points are at $-1$, $0$ and $1$. The solution is guaranteed to exist on $(-\infty, -1)$.

11. a) For $t = 0$, $y' = -1$ and $y'' = 1$. Only Graph B is consistent with that.
   b) For $t = 0$, $y' = -1$ and $y'' = -1$. Only Graph D is consistent with that.
   c) For $t = 0$, $y' = 1$ and $y'' = 1$. Only Graph A is consistent with that.
   d) For $t = 0$, $y' = 1$ and $y'' = -1$. Only Graph C is consistent with that.

13. a) $\omega$ decreases with increasing $\rho$, so the drum submerged less (1) will oscillate faster.
   b) $\omega$ decreases as a function of $L$, so the shorter drum (1) will oscillate faster.
Solutions 3.2

2. a) Yes.
   b) 
   \[ W = \begin{vmatrix} 2e^t & e^{-t+3} \\ 2e^t & -e^{-t+3} \end{vmatrix} = -4e^3 \neq 0, \]
   so they are a fundamental set.
   c) The general solution is \( y = 2c_1 e^t + c_2 e^{-t+3} \). To satisfy the initial conditions, we need 
   \[ 2e^{-1}c_1 + e^4c_2 = 1, \quad 2e^{-1}c_1 - e^4c_2 = 0. \]
   This yields \( c_1 = e/4, \ c_2 = e^{-4}/2 \).

4. a) Yes.
   b) 
   \[ W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1 \neq 0, \]
   so they are a fundamental set.
   c) The general solution is \( y = c_1 \cos t + c_2 \sin t \). To satisfy the initial conditions, we need 
   \( c_2 = 1, \quad -c_1 = 1 \).

10. a) Yes.
   b) 
   \[ W = \begin{vmatrix} \ln t & \ln 3 \\ 1/t & 0 \end{vmatrix} = -\frac{\ln 3}{t} \neq 0, \]
   so they are a fundamental set.
   c) The general solution is \( y = c_1 \ln t + c_2 \ln 3 \). To satisfy the initial conditions, we need 
   \( c_2 \ln 3 = 0, \quad c_1 = 3 \).

15. a) Yes.
   b) 
   \[ W = \begin{vmatrix} e^{-t/2} & te^{-t/2} \\ -\frac{1}{2}e^{-t/2} & (1 - \frac{1}{2})e^{-t/2} \end{vmatrix} = e^{-t}. \]
   c) The general solution is 
   \[ y = c_1 e^{-t/2} + c_2 te^{-t/2}. \]
   We have 
   \[ y(1) = c_1/\sqrt{e} + c_2/\sqrt{e} = 1, \]
   \( 1 \)
\[ y'(1) = -\frac{c_1}{2\sqrt{e}} + \frac{c_2}{2\sqrt{e}} = 0, \]
hence \( c_1 = c_2 = \sqrt{e}/2. \)

16. 
\[ \sin(2t + \frac{\pi}{4}) = \sin(2t)\cos\frac{\pi}{4} + \cos(2t)\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\sin(2t) + \frac{\sqrt{2}}{4}(2\cos(2t)). \]

19. We find 
\[ y''_1 + \alpha y'_1 + \beta y_1 = (9 + 3\alpha + \beta)e^{3t} = 0, \]
so we must have 
\[ 9 + 3\alpha + \beta = 0. \]

Proceeding analogously with \( y_2 \), we find
\[ 9 - 3\alpha + \beta = 0. \]
Consequently, \( \alpha = 0 \) and \( \beta = -9. \)
Solutions 3.3

2. a) The characteristic equation is
   \[ \lambda^2 - \frac{1}{4} = 0, \]
   leading to \( \lambda = \pm 1/2 \). The general solution is
   \[ y = c_1 e^{t/2} + c_2 e^{-t/2}. \]

   b) \[ y(2) = c_1 e + c_2 e^{-1} = 1, \quad y'(2) = \frac{e}{2}c_1 - \frac{1}{2}c_2 = 0, \]
   hence \( c_1 = 1/(2e) \), \( c_2 = e/2 \).

   c) \( y(t) \) approaches \( +\infty \) at both ends.

7. a) The characteristic equation is
   \[ \lambda^2 + 5\lambda + 6 = 0, \]
   leading to \( \lambda = -3, -2 \). The general solution is
   \[ y = c_1 e^{-3t} + c_2 e^{-2t}. \]

   b) \[ y(0) = c_1 + c_2 = 1, \quad y'(0) = -3c_1 - 2c_2 = -1, \]
   hence \( c_1 = -1, c_2 = 2 \).

   c) \( \lim_{t \to -\infty} y(t) = -\infty, \lim_{t \to -\infty} y(t) = 0. \)

13. a) The characteristic equation is
   \[ \lambda^2 + 4\lambda + 2 = 0, \]
   leading to \( \lambda = -2 \pm \sqrt{2} \). The general solution is
   \[ y = c_1 e^{-2-\sqrt{2}t} + c_2 e^{-2+\sqrt{2}t}. \]

   b) \[ y(0) = c_1 + c_2 = 0, \quad y'(0) = (-2 - \sqrt{2})c_1 + (-2 + \sqrt{2})c_2 = 4, \]
   hence \( c_1 = -\sqrt{2}, c_2 = \sqrt{2} \).

   c) \( \lim_{t \to -\infty} y(t) = -\infty, \lim_{t \to -\infty} y(t) = 0. \)

18. The roots of the characteristic equation are \(-2\) and \(0\) for part a, \(1/2\) and \(1/3\) for part b, and \(\pm 1\) for part c. All solutions other than the zero solution will go to infinity for the equation in b, and only Graph B is consistent with that. Moreover, the solution in Graph C has a nonzero limit as \( t \to \infty \), which is possible only for equation a.