Solutions 2.2

1. We have \( P(t) = 3t \), so the general solution is \( y = C \exp(-3t) \). The initial condition leads to \( C = -3 \).

4. We have \( p(t) = -4/t \), so \( P(t) = -4 \ln t \), and the general solution is \( y = Ct^4 \). The initial condition yields \( C = 1 \).

6. The integrating factor is \( e^{-2t} \). Hence we can put the equation in the form

\[
e^{-2t}(y' - 2y) = \frac{d}{dt}(e^{-2t}y) = e^t.
\]

Integration leads to

\[
e^{-2t}y = e^t + C,
\]

i.e.

\[
y = e^{3t} + Ce^{2t}.
\]

Finally the initial condition yields

\[
y(0) = 1 + C = 3,
\]

i.e. \( C = 2 \).

12. We have \( P(t) = t - \cos t \), so the general solution is \( y(t) = C \exp(-t + \cos t) \).

15. The standard form of the equation is

\[
y' - 3(t^2 + 1)y = 0.
\]

Hence \( p(t) = -3(t^2 + 1) \), \( P(t) = -t^3 - 3t \), and the general solution is \( y = C \exp(t^3 + 3t) \).

18. The integrating factor is \( e^{2t} \). Hence we can put the equation in the form

\[
e^{2t}(y' + 2y) = \frac{d}{dt}(e^{2t}y) = e^t.
\]

Integration leads to

\[
e^{2t}y = e^t + C,
\]

i.e.

\[
y = e^{-t} + Ce^{-2t}.
\]

19. The integrating factor is \( e^{2t} \). Hence we can put the equation in the form

\[
e^{2t}(y' + 2y) = \frac{d}{dt}(e^{2t}y) = 1.
\]
Integration leads to
\[ e^{2t}y = t + C, \]
i.e.
\[ y = te^{-2t} + Ce^{-2t}. \]

21. First put the equation in standard form
\[ y' + \frac{2}{t}y = t. \]

The integrating factor is \( t^2 \). Hence we can put the equation in the form
\[ t^2y' + 2ty = \frac{d}{dt}(t^2y) = t^3. \]

Integration leads to
\[ t^2y = \frac{t^4}{4} + C, \]
i.e.
\[ y = \frac{t^2}{4} + Ct^{-2}. \]

26. The general solution of the differential equation is \( y = Ce^{-\alpha t} \) and the initial condition leads to \( y = y_0e^{-\alpha t} \). To pass through the two given points, we must satisfy
\[ 4 = y_0e^{-\alpha}, \quad 1 = y_0e^{-3\alpha}. \]

By dividing the two equations, we get
\[ e^{-2\alpha} = \frac{1}{4}, \]
leaving to \( \alpha = \ln 2 \). Inserting this into one of the two equations yields \( y_0 = 8 \).

28. Note that \( y'/y \) is \(-1/2\) for Problem a, \(-\cos(2t)\) for Problem b, \((1 - \cos(2t))/10\) for Problem c and \(1/10\) for Problem d. Since \( y \) is positive on all the graphs, \( y' \) must be negative for Problem a, oscillating in sign for Problem b, positive with zeros when \( t \) is a multiple of \( \pi \) for part c, and positive for part d. Consequently, Graph 1 corresponds to c, Graph 2 to a, Graph 3 to d, and Graph 4 to b.

29. a) We find the new equation
\[ \frac{dB(c)}{dc} = -kB(c), \quad B(0) = -A^*. \]
b) The solution for part a is

\[ B(c) = -A^* e^{-kc}, \]

and hence

\[ A(c) = A^*(1 - e^{-kc}). \]

For positive \( c \), this is always between 0 and \( A^* \).

c) We want to find \( c \) such that

\[ 1 - e^{-kc} = 0.95. \]

Hence \( e^{-kc} = 0.05 = 1/20 \), and \( c = \ln 20 / k \).

32. We have

\[ y'(t) = 2tCe^{t^2}. \]

We therefore want

\[ 2tCe^{t^2} + p(t)(Ce^{t^2} + 2) = g(t) \]

for every \( t \) and \( C \). Comparing coefficients of \( C \), we find

\[ 2te^{t^2} + p(t)e^{t^2} = 0, \]

i.e. \( p(t) = -2t \). The remaining terms then yield

\[ 2p(t) = g(t), \]

i.e. \( g(t) = -4t \).

36. The integrating factor is \( \exp(t + \sin t) \), and we can put the equation in the form

\[ \frac{d}{dt} (\exp(t + \sin t)y(t)) = \exp(t + \sin t)(1 + \cos t). \]

Integration yields

\[ \exp(t + \sin t)y(t) = \exp(t + \sin t) + C, \]

i.e.

\[ y(t) = 1 + C \exp(-t - \sin t). \]

Consequently, \( y(t) \to 1 \) as \( t \to \infty \). The initial condition is

\[ y(0) = 1 + C = 3, \]
hence $C = 2$.

40. The integrating factor is $t$. For $g(t) = 3t$, we can therefore put the equation in the form

$$ty' + t = \frac{d}{dt}(ty) = 3t^2.$$  

Integration yields

$$ty = t^3 + C,$$

i.e. $y = t^2 + C/t$.

For $g(t) = 0$, we find $y = C/t$ in an analogous fashion.

Consequently, we have $y = t^2 + C_1/t$ for $1 \leq t \leq 2$ and $y = C_2/t$ for $2 \leq t \leq 3$.

The initial condition at $t = 1$ leads to

$$1 = 1 + C_1,$$

and continuity at $t = 2$ leads to

$$4 + C_1/2 = C_2/2.$$

These two equations yield $C_1 = 0$ and $C_2 = 8$. 
Solutions 2.3

1. Let \( Q(t) \) be the amount of salt in the tank measured in lb, and let \( t \) be measured in minutes. The equation governing \( Q \) is

\[
Q' = 0.6 - \frac{3}{100}Q, \quad Q(0) = 0.
\]

The general solution of the differential equation is

\[
Q(t) = 20 + C \exp\left(-\frac{3}{100}t\right),
\]

and the initial condition yields \( C = -20 \).

a) At \( t = 10 \), we have

\[
Q(10) = 20 - 20 \exp\left(-\frac{3}{10}\right) = 5.184.
\]

b) As \( t \to \infty \), \( Q(t) \to 20 \). The limiting concentration is 20 lb/100 gal or 0.2 lb/gal, same as the concentration at inflow.

4. Let the rate of pumping be \( r \) gal/min. Then the equation governing \( Q \) is

\[
Q' = 0.1r - \frac{r}{200}Q, \quad Q(0) = 5.
\]

The general solution of the differential equation is

\[
Q(t) = 20 + C \exp\left(-\frac{rt}{200}\right).
\]

The initial condition yields \( C = -15 \).

a) We want

\[
15 = Q(20) = 20 - 15e^{-r/10},
\]

i.e.

\[
e^{-r/10} = \frac{1}{3}.
\]

This yields \( r = 10 \ln(3) = 10.99 \).

b) Regardless of the value of \( r \), \( Q(t) \) will never exceed 20.

11.a) We have \( c(t) = Q(t)/500 \), and the concentration at inflow is \( \alpha c(t) \).

The basic conservation principle implies that

\[
Q' = 15\alpha c(t) - 15c(t) = 15(\alpha - 1)Q(t)/500.
\]
The solution is
\[ Q(t) = Q(0) \exp(15(\alpha - 1)t/500). \]
b) We want \( Q(180) = Q(0)/100 \), i.e.
\[ \frac{1}{100} = \exp(5.4(\alpha - 1)), \]
which leads to
\[ 1 - \alpha = 0.8528, \quad \alpha = 0.1472. \]

13. a) With \( Q \) measured in lb and \( t \) measured in hours, we have
\[ Q'_A = -Q_A/500, \quad Q'_B = Q_A/500 - Q_B/200. \]
Note that the salt going into pond B is the same salt that is leaving pond A. The initial conditions are
\[ Q_A(0) = 1000, \quad Q_B(0) = 0. \]
b) We find \( Q_A(t) = 1000 \exp(-t/500). \) In the equation for \( Q_B \) this leads to
\[ Q'_B = 2e^{-t/500} - \frac{Q_B}{200}, \quad Q_B(0) = 0. \]
The solution of this is
\[ Q_B(t) = \frac{2000}{3}(e^{-t/500} - e^{-t/200}). \]
c) We have
\[ Q'_B = \frac{2000}{3}(-\frac{1}{500}e^{-t/500} + \frac{1}{200}e^{-t/200}). \]
This is zero when
\[ \frac{1}{500}e^{-t/500} = \frac{1}{200}e^{-t/200}, \]
i.e.
\[ e^{3t/1000} = \frac{5}{2}. \]
This yields \( t = \frac{1000}{3} \ln(2.5) = 305.43; \) at this time \( Q_B(t) = 80 \times 20^{3/3}. \)
d) We want \( Q_A(t) \leq 1/2 \) and \( Q_B(t) \leq 1/5. \) We obtain \( Q_A(t) = 1/2 \) when
\[ t = 500 \ln(2000) = 3800.45. \]
We have \( Q_B(t) = 1/5 \) when 
\[
e^{-t/500} - e^{-t/200} = \frac{3}{10000}.
\]
We cannot solve this in closed form, but a numerical solution leads to \( t = 4055.861 \). Another possibility is an approximate solution. For large \( t \), \( \exp(-t/200) \) will be much less than \( \exp(-t/500) \). So we have the approximate equation
\[
\exp(-t/500) = \frac{3}{10000},
\]
leading to \( t = 500 \ln(10000/3) = 4055.864 \), indistinguishable from the exact solution at any realistic level of accuracy.

14. a) The amount of solution remains constant, because the inflow and outflow rates are the same.

b) You expect that the concentration will approach the inflow concentration. Since only the flow rate, but not the concentration varies, you expect an equilibrium value.

c) The initial value problem is
\[
Q' = 0.5(3 + \sin t) - (3 + \sin t)\frac{Q(t)}{200}, \quad Q(0) = 10.
\]

d) We write the equation in the form
\[
Q' + \frac{3 + \sin t}{200}Q(t) = \frac{3 + \sin t}{2}.
\]
The integrating factor is 
\[
\exp\left(\frac{3t - \cos t}{200}\right).
\]
We multiply and obtain
\[
\frac{d}{dt}(\exp\left(\frac{3t - \cos t}{200}\right)Q(t)) = \frac{3 + \sin t}{2} \exp\left(\frac{3t - \cos t}{200}\right).
\]
Integration yields
\[
(\exp\left(\frac{3t - \cos t}{200}\right)Q(t)) = 100 \exp\left(\frac{3t - \cos t}{200}\right) + C,
\]
hence
\[
Q(t) = 100 + C \exp\left(\frac{3t - 3\cos t}{200}\right).
\]
As $t \to \infty$, $Q(t) \to 100$. The value of the constant is determined by

$$Q(0) = 100 + Ce^{1/200} = 10,$$

leading to $C = -90e^{-1/200}$. 
Solutions 2.4

8. The amount of the substance must be given by \( Ce^{-kt} \) for some constants \( C \) and \( k \). If \( t \) is measured in days, we are told that \( Ce^{-30k} = 100 \) and \( Ce^{-120k} = 30 \). Dividing the two equations, we find \( e^{-90k} = 3/10 \), i.e. \( k = \frac{1}{90} \ln(10/3) \sim 0.01338 \). We then obtain \( C = 100e^{30k} \sim 149.38 \). This answers part a. The half life is \( \tau = \ln 2/k \sim 51.83 \) days. The time until 1% remains is \( \ln 100/k \sim 344.3 \) days.

13. a) We are told that
\[ \ln 2/k = 5730, \]
hence \( k = \ln 2/5730 \sim 0.00012097 \). Next we are told that
\[ e^{-kt} = 0.3, \]
i.e.
\[ t = \frac{1}{k} \ln(10/3) \sim 9953 \]
years.

b) We can repeat the calculation in part a assuming \( \tau = 5700 \) and \( \tau = 5760 \), respectively. The corresponding values of \( k \) are 0.0001216 and 0.00012034. The ages obtained are 9901 and 10005 years. Hence the uncertainty is approximately 50 years.

c) We have \( Q(60000)/Q(0) = \exp(-60000k) \sim 0.000704 \). Hence less than one thousandth of the original amount is still remaining.

15. We have \( k = \ln 2/8 = 0.0866 \). We are told that \( Q(0)e^{-3k} = 30 \), hence
\[ Q(0) = 30e^{3k} = 38.9 \]
 micrograms.