An Improvement in the Penalty Method for the Power-Law Stokes Problems

Jeff Borggaard [a,1], Traian Iliescu [b,2], John Paul Roop [c,*,3],

[a] *Department of Mathematics, Virginia Tech, Blacksburg, VA 24061-0123, U.S.A.*

[b] *Department of Mathematics, Virginia Tech, Blacksburg, VA 24061-0123, U.S.A.*

[c] *Department of Mathematics, North Carolina A&T State University, Greensboro, NC ?????, U.S.A.*

Abstract

In this article, we show theoretically and numerically that a *linear* penalty function approximation to a power-law Stokes problem yields a higher order accuracy over the known nonlinear penalty method. The *linear* penalty function method is shown to satisfy a *linear* order of approximation for finite element approximations on a fixed grid.

Key words: Penalty method, power-law Stokes, p-Laplacian, large eddy simulation, Smagorinsky model.
PACS:

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* Corresponding author.
Email addresses: borggajt@vt.edu (Jeff Borggaard), iliescu@math.vt.edu (Traian Iliescu), jroop@vt.edu (John Paul Roop).
URLs: www.????? (Jeff Borggaard), www.math.vt.edu/people/iliescu (Traian Iliescu), www.????? (John Paul Roop).

[1] Supported by the National Science Foundation and the Air Force Office for Sponsored Research.
[2] Partially supported by AFOSR grants F49620-03-1-0243 and FA9550-05-1-0449 and NSF grants DMS-0209309, DMS-0322852, and DMS-0513542.
[3] Supported by the National Science Foundation and the Air Force Office for Sponsored Research.

1 Introduction

In this article we analyze mathematically and numerically an alternative penalty method for the solution of stationary *power-law Stokes problem*

$$\begin{cases} -\nu\,\nabla\cdot(|\nabla \boldsymbol{u}|^{r-2}\nabla \boldsymbol{u}) + \nabla p = \mathbf{f}, \text{ in } \Omega, \\ \nabla\cdot \boldsymbol{u} = 0, \text{ in } \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^d, d = 2,3$ is the computational domain, ν the kinematic viscosity, $\boldsymbol{u}$ the velocity vector, p the pressure, and $\mathbf{f}$ the body force. The power-law Stokes equations (1) have been used as a mathematical model in numerous applications of non-Newtonian flows: in chemical engineering [8], design of extrusion of dies [19], the study of litosphere [11], and other geophysical applications [26]. Equally important, the power-law Stokes equations (1) can be thought of as a simplified, linearized setting for the *Smagorinsky model* [25], one of the most popular models in the *large eddy simulation* of turbulent flows [7,23].

The *penalty method* for the power-law Stokes equations (1)

$$\begin{cases} -\nu\,\nabla\cdot\nabla \boldsymbol{u}^\varepsilon + \nabla p^\varepsilon = \mathbf{f}, \text{ in } \Omega, \\ \varepsilon\, p^\varepsilon + \nabla\cdot \boldsymbol{u}^\varepsilon = 0, \text{ in } \Omega, \end{cases} \tag{2}$$

where ε is a small penalty parameter (typically $\varepsilon \sim 10^{-4} - 10^{-3}$ in practical calculations), decouples the calculation of velocity $\boldsymbol{u}^\varepsilon$ and pressure p^ε, and thus decreases the computational cost. The penalty method has been used extensively in the numerical simulation of fluid flows [10,13,21] because of its computational efficiency. The mathematical and numerical analyses for the penalty method applied to the Navier-Stokes equations have been also provided [27,14,24]. Mathematical support for the penalty method applied to the power-law Stokes equations (1) was provided in a series of papers by Lefton and Wei [16,28,17]. The authors have considered a nonlinear penalty method

$$\begin{cases} -\nu\,\nabla\cdot(|\nabla \boldsymbol{u}^\varepsilon|^{r-2}\nabla \boldsymbol{u}^\varepsilon) + \nabla p^\varepsilon = \mathbf{f}, \text{ in } \Omega, \\ \varepsilon\, p^\varepsilon + (|\nabla\cdot \boldsymbol{u}^\varepsilon|^{r-2})\nabla\cdot \boldsymbol{u}^\varepsilon = 0, \text{ in } \Omega \end{cases} \tag{3}$$

and proved error estimates of the form

$$\|\boldsymbol{u} - \boldsymbol{u}^\varepsilon\|_{1,r} \le \begin{cases} C\,\varepsilon^{1/(r-1)}, \text{ if } r > 2, \\ C\,\epsilon^{1/(3-r)}, \text{ if } 1 < r < 2, \end{cases} \tag{4}$$

where C is a generic constant independent of ε. Also, similar estimates were derived for the stationary power-law Navier-Stokes equations [28]. Note that the penalty method applied to the Navier-Stokes equations yields higher order, linear error estimates [13,21]

$$\|\boldsymbol{u} - \boldsymbol{u}^{\varepsilon}\|_1 \leq C\,\varepsilon. \tag{5}$$

It is the goal of this paper to show that, in fact, the finite element discretization of a *linear* penalty function method applied to the power-law Stokes problem

$$\begin{cases} -\nu\,\nabla\cdot(|\nabla\boldsymbol{u}^{\varepsilon}|^{r-2}\nabla\boldsymbol{u}^{\varepsilon}) + \nabla p^{\varepsilon} = \mathbf{f}, \text{ in } \Omega, \\ \varepsilon\,p^{\varepsilon} + \nabla\cdot\boldsymbol{u}^{\varepsilon} = 0, \text{ in } \Omega \end{cases} \tag{6}$$

satisfies the *linear* error estimate

$$\|\boldsymbol{u}_h - \boldsymbol{u}_h^{\varepsilon}\|_{1,r} \leq C(h)\,\varepsilon, \tag{7}$$

where $\boldsymbol{u}_h, \boldsymbol{u}_h^{\varepsilon}$ are finite element approximations of the continuous solutions $\boldsymbol{u}, \boldsymbol{u}^{\varepsilon}$ to (2) and (6), respectively, and $C(h)$ is a generic constant which, for a given finite element approximation, may depend on the mesh parameter h but not on ε. This is a significant improvement over the nonlinear penalty method (3), since the computational efficiency is the main reason for using the penalty method in practical calculations. We also note that a similar linear penalty method has been used in [22] to approximate power-law flows. The authors have achieved good numerical results, but have not provided any mathematical support for their approach.

The paper is organized as follows. In Section 2, we introduce the notation and functional spaces used in the paper. In Section 3, we introduce the improved linear penalty method for the power-law Stokes flows (6) and prove the existence and uniqueness of solutions. In Section 4, we prove improved error estimates with respect to the penalty parameter ε for the continuous linear penalty method (6). In Section 5, we prove existence and uniqueness of solutions as well as improved linear error estimates with respect to ε for the finite element discretization of the linear penalty method (6). In Section 6, we present numerical experiments for the power-law Stokes problem (2), which illustrate the improved rate of convergence for the linear penalty method over the nonlinear penalty method. Finally, in Section 7 we present conclusions and directions of future research.

2 Notation and Mathematical Setting

We present below some of the notations and functional analysis results that will be frequently used in the paper.

Let $L^r(\Omega), W^{k,r}(\Omega)$, and $W_0^{k,r}(\Omega), 1 < r < \infty, k = 0, 1, 2, \ldots$ denote the usual Sobolev spaces [1]. Let $\|\cdot\|$ denote the norm on $L^2(\Omega)$, $\|\cdot\|_r = \|\cdot\|_{0,r}$ the norm on $L^r(\Omega)$, and $\|\cdot\|_{k,r}$ the norm on $W^{k,r}(\Omega)$. Let $(\cdot,\cdot)$ denote the scalar product in $L^2(\Omega)$. The vector spaces and vector functions will be indicated by boldface type letters.

For $1 < r < \infty$, let r' denote the conjugate of r: $\left(\dfrac{1}{r} + \dfrac{1}{r'} = 1, \quad \text{i.e.} \quad r' = \dfrac{r}{r-1}\right)$. Let $W^{-k,r'}(\Omega)$ denote the dual space of $W_0^{k,r}(\Omega)$, and $\|\cdot\|_{-k,r'}$ the norm on $W^{-k,r'}(\Omega)$. Let $L_0^r(\Omega) := \left\{q \in L^r(\Omega) : \int_\Omega q \, d\mathbf{x} = 0\right\}$.

We will also use the strong monotonicity and Lipschitz continuity of the r-Laplacian [18,20]:

$$\begin{aligned}
&\left(|\nabla \boldsymbol{u}_1|^{r-2} \nabla \boldsymbol{u}_1, \nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)\right) - \left(|\nabla \boldsymbol{u}_2|^{r-2} \nabla \boldsymbol{u}_2, \nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)\right) \\
&\qquad \geq C \, \|\nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|_{0,r}^r \qquad\qquad \forall\, \boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbf{W}^{1,r}(\Omega), \qquad (8) \\
&\left(|\nabla \boldsymbol{u}_1|^{r-2} \nabla \boldsymbol{u}_1, \nabla \boldsymbol{v}\right) - \left(|\nabla \boldsymbol{u}_2|^{r-2} \nabla \boldsymbol{u}_2, \nabla \boldsymbol{v}\right) \\
&\qquad \leq C \, \|\nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|_{0,r}^{r-1} \, \|\nabla \boldsymbol{v}\|_{0,r} \qquad \forall\, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v} \in \mathbf{W}^{1,r}(\Omega), \quad (9)
\end{aligned}$$

where C is a generic constant depending on d, r, and Ω, but not on $\boldsymbol{u}_1, \boldsymbol{u}_2$ or $\boldsymbol{v}$.

In the sequel, we will also assume $r \geq 2$, although some of our results carry over to the $r < 2$ case.

3 The Penalty Method

In this section, we will introduce the improved, linear penalty method for the stationary power-law Stokes problem (1). We will also prove that there exists a unique solution to the linear penalty method.

Let $\mathbf{X} := \mathbf{W}_0^{1,r}(\Omega)$ and $Q := L_0^{r'}(\Omega)$.

The *mixed weak formulation* of the stationary power-law Stokes problem (1) reads

$$
\begin{cases}
\nu\left(|\nabla \boldsymbol{u}|^{r-2}\,\nabla \boldsymbol{u}, \nabla \boldsymbol{v}\right) - (p, \nabla\cdot \boldsymbol{v}) = (\mathbf{f}, \boldsymbol{v}), & \forall\, \boldsymbol{v} \in \mathbf{X}, \\
(\nabla\cdot \boldsymbol{u}, q) = 0, & \forall\, q \in Q,
\end{cases}
\tag{10}
$$

The existence and uniqueness of solutions $(\boldsymbol{u}, p) \in \mathbf{X} \times Q$ to (10) was studied by Baranger and Najib [4] and Barrett and Liu [6].

The *mixed weak formulation* of the linear penalty method applied to the stationary power-law Stokes problem (6) reads

$$
\begin{cases}
\nu\left(|\nabla \boldsymbol{u}^\varepsilon|^{r-2}\,\nabla \boldsymbol{u}^\varepsilon, \nabla \boldsymbol{v}\right) - (p^\varepsilon, \nabla\cdot \boldsymbol{v}) = (\mathbf{f}, \boldsymbol{v}), & \forall\, \boldsymbol{v} \in \mathbf{X}, \\
(\nabla\cdot \boldsymbol{u}^\varepsilon, q) + \varepsilon(p^\varepsilon, q) = 0, & \forall\, q \in Q,
\end{cases}
\tag{11}
$$

To study the existence and uniqueness of solutions $(\boldsymbol{u}^\varepsilon, p^\varepsilon) \in \mathbf{X} \times Q$ to (11), we follow [16,6,9].

First, define the functional $J_\varepsilon : \mathbf{X} \to \mathbb{R}$ given by

$$
J_\varepsilon(\boldsymbol{u}) := \frac{\nu}{r}\,\|\nabla \boldsymbol{u}\|_{0,r}^r + \frac{1}{2\,\varepsilon}\,\|\nabla\cdot \boldsymbol{u}\|^2 - (\mathbf{f}, \boldsymbol{u}) \qquad \forall \boldsymbol{u} \in \mathbf{X}. \tag{12}
$$

Note that J_ε is well defined. Indeed, since $\boldsymbol{u} \in \mathbf{X} = \mathbf{W}_0^{1,r}(\Omega)$ and $r \geq 2$, it follows that $\nabla\cdot \boldsymbol{u} \in L^2(\Omega)$.

Let $\langle\cdot,\cdot\rangle$ denote the duality pairing between $\mathbf{X}$ and $\mathbf{X}^*$. It is a straightforward calculation to check that $J_\varepsilon(\cdot)$ is Gâteaux differentiable on $\mathbf{X}$. Indeed,

$$
\langle J_\varepsilon'(\boldsymbol{u}), \boldsymbol{v}\rangle = \frac{dJ_\varepsilon}{dt}(\boldsymbol{u} + t\,\boldsymbol{v}) = \langle A\boldsymbol{u}, \boldsymbol{v}\rangle - \langle \mathbf{f}, \boldsymbol{v}\rangle, \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbf{X}, \tag{13}
$$

where $A : \mathbf{X} \to \mathbf{X}^*$ is such that

$$
\langle A\boldsymbol{u}, \boldsymbol{v}\rangle := \nu\left(|\nabla \boldsymbol{u}|^{r-2}\,\nabla \boldsymbol{u}, \nabla \boldsymbol{v}\right) + \frac{1}{\varepsilon}\,(\nabla\cdot \boldsymbol{u}, \nabla\cdot \boldsymbol{v}). \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbf{X}, \tag{14}
$$

Moreover, $J_\varepsilon'(\cdot)$ is strictly monotone. Indeed,

$$
\begin{aligned}
\langle J_\varepsilon'(\boldsymbol{w}_1) - J_\varepsilon'(\boldsymbol{w}_2), \boldsymbol{w}_1 - \boldsymbol{w}_2\rangle &= \langle A\boldsymbol{w}_1 - A\boldsymbol{w}_2, \boldsymbol{w}_1 - \boldsymbol{w}_2\rangle \\
&= \nu\left(|\nabla \boldsymbol{w}_1|^{r-2}\,\nabla \boldsymbol{w}_1 - |\nabla \boldsymbol{w}_2|^{r-2}\,\nabla \boldsymbol{w}_2, \nabla(\boldsymbol{w}_1 - \boldsymbol{w}_2)\right) \\
&+ \frac{1}{\varepsilon}\,(\nabla\cdot(\boldsymbol{w}_1 - \boldsymbol{w}_2), \nabla\cdot(\boldsymbol{w}_1 - \boldsymbol{w}_2)) \\
&\geq C\,\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{1,r}^r\ ,
\end{aligned}
\tag{15}
$$

where in the last inequality we used the strong monotonicity of the r-Laplacian (8).

Therefore, $J_\varepsilon(\cdot)$ is strictly convex. (See Section 3 in [4].) Furthermore, $J_\varepsilon(\cdot)$ is coercive on $\mathbf{X}$.

Thus, we have proved the following result:

Lemma 3.1 *There exists a unique solution to the minimization problem*

$$\min_{\boldsymbol{v}\in\mathbf{X}} J_\varepsilon(\boldsymbol{v}). \tag{16}$$

Remark 3.1 *The minimization problem* (16) *is the variational formulation of* (6)*, the linear penalty method for the power-law Stokes problem.*

We will prove now that the existence and uniqueness of the solution to the variational formulation (16) implies the existence and uniqueness of solutions to the mixed weak formulation of the power-law Stokes problem (11). First, we note that the Euler-Lagrange equation of (16) is

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = (\mathbf{f}, \boldsymbol{v}), \qquad \forall\, \boldsymbol{v} \in \mathbf{X}. \tag{17}$$

Let $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ be a solution of (11). Then, by using (11_2) with

$$q := \frac{1}{|\Omega|}\int_\Omega \nabla\cdot\boldsymbol{v}\, d\mathbf{x} - \nabla\cdot\boldsymbol{v} \in L^r \subset L^{r'} \quad (\text{since } r \geq 2), \tag{18}$$

we obtain

$$\langle A\boldsymbol{u}^\varepsilon, \boldsymbol{v}\rangle = (\mathbf{f}, \boldsymbol{v}), \qquad \forall\, \boldsymbol{v} \in \mathbf{X}, \tag{19}$$

which implies that $\boldsymbol{u}^\varepsilon$ is a solution to (17).

Conversely, let $\boldsymbol{u}$ be a solution to (17). Pick $\boldsymbol{u}^\varepsilon := \boldsymbol{u}$ and

$$p^\varepsilon := \frac{1}{\varepsilon\,|\Omega|}\int_\Omega \nabla\cdot\boldsymbol{u}^\varepsilon\, d\mathbf{x} - \frac{1}{\varepsilon}(\nabla\cdot\boldsymbol{u}^\varepsilon). \tag{20}$$

It is a simple calculation to check that $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ is a solution to (11).

In order to prove the uniqueness of p^ε, however, we require the Ladyzhenskaya-Babuska-Brezzi (LBB) inf-sup condition [12,13]. Amrouche and Girault [2] proved the following result:

Theorem 3.1 *Let Ω be a bounded, connected, Lipschitz continuous domain in $\mathbb{R}^d$ and let r be any real number with $1 < r < \infty$, and r' its conjugate. There exists a constant $\beta > 0$ such that*

$$0 < \beta \leq \inf_{q \in L_0^{r'}(\Omega)} \sup_{\boldsymbol{v} \in \mathbf{W}_0^{1,r}(\Omega)} \frac{(\nabla \cdot \boldsymbol{v}, q)}{\|q\|_{0,r'} \, \|\boldsymbol{v}\|_{1,r}}. \tag{21}$$

In the sequel, we will assume that the domain Ω satisfies the conditions in Theorem 3.1, and thus the LBB condition is satisfied. We are now in the position to prove the uniqueness of p^ε. Assume that $p_1^\varepsilon, p_2^\varepsilon \in Q$ satisfy (11). We know that $\boldsymbol{u}^\varepsilon$ is unique (Lemma 3.1). Thus, (11_2) for p_1^ε and p_2^ε yields

$$\varepsilon\,(p_1^\varepsilon - p_2^\varepsilon, \nabla \cdot \boldsymbol{v}) = 0 \qquad \forall\, \boldsymbol{v} \in \mathbf{W}_0^{1,r}. \tag{22}$$

By picking $q := p_1^\varepsilon - p_2^\varepsilon$ in (21), we get

$$\sup_{\boldsymbol{v} \in \mathbf{W}_0^{1,r}(\Omega)} \frac{(\nabla \cdot \boldsymbol{v}, p_1^\varepsilon - p_2^\varepsilon)}{\|\boldsymbol{v}\|_{1,r}} \geq \beta\, \|p_1^\varepsilon - p_2^\varepsilon\|_{0,r'}. \tag{23}$$

Thus, (22) and (23) imply $0 > 0$. This contradiction proves the uniqueness of p^ε.

Thus, since $\varepsilon > 0$, we have proved the following result:

Theorem 3.2 *There exists a unique solution $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ to the mixed weak formulation of the linear penalty method for the power-law Stokes problem* (11). *Moreover, the unique pressure $p^\varepsilon in L_0^r(\Omega)$ is given by the following formula*

$$p^\varepsilon = \frac{1}{\varepsilon\,|\Omega|} \int_\Omega \nabla \cdot \boldsymbol{u}^\varepsilon \, d\mathbf{x} - \frac{1}{\varepsilon}\,(\nabla \cdot \boldsymbol{u}^\varepsilon). \tag{24}$$

4 Error Analysis for the Continuous Linear Penalty Method

In this section, we prove that $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$, the solution to the continuous linear penalty method (11) converges to (u, p), the solution to the power-law Stokes problem (10) strongly with respect to ε.

We start by proving an *a priori* bound for $\boldsymbol{u}$ and $\boldsymbol{u}^\varepsilon$.

Lemma 4.1 *Let $(\boldsymbol{u}, p)$ be the solution of (10) and $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ the solution of (11). Then*

$$\|\boldsymbol{u}\|_{1,r} \leq C \qquad \textit{and} \qquad \|\boldsymbol{u}^\varepsilon\|_{1,r} \leq C, \tag{25}$$

where C is a generic constant depending only on r, Ω, and $\mathbf{f}$.

Proof. The *a priori* bound for $\boldsymbol{u}$ was proved in [16]. Setting $\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{u}^\varepsilon$ in (17), and using the strong monotonicity of the r-Laplacian (8) and the Cauchy-Scwartz inequality, yields

$$\|\boldsymbol{u}^\varepsilon\|_{1,r} \leq \left(\frac{1}{C}\,\|\mathbf{f}\|_{-1,r'}\right)^{\frac{1}{r-1}}, \tag{26}$$

which proves the lemma. □

The next theorem proves the strong convergence of $\boldsymbol{u}^\varepsilon$ to $\boldsymbol{u}$ as $\varepsilon \to 0$ in the $\mathbf{W}^{1,r}(\Omega)$ norm, provided that p is contained in $L^2(\Omega)$.

Theorem 4.1 *Let $(\boldsymbol{u}, p)$ solve (10) and $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ solve (11). Then, provided that $p \in L^2(\Omega)$, $\boldsymbol{u}^\varepsilon \to \boldsymbol{u}$ as $\varepsilon \to 0$ strongly in the $W^{1,r}(\Omega)$ norm. Specifically,*

$$\|\boldsymbol{u} - \boldsymbol{u}^\varepsilon\|_{1,r} \leq C\,\varepsilon^{1/r}. \tag{27}$$

where C is a generic constant depending on ν, but not on ε.

Proof. First, by subtracting (11_1) from (10_1), we get

$$\nu\left(|\nabla\boldsymbol{u}|^{r-2}\nabla\boldsymbol{u}, \nabla\boldsymbol{v}\right) - \nu\left(|\nabla\boldsymbol{u}^\varepsilon|^{r-2}\nabla\boldsymbol{u}^\varepsilon, \nabla\boldsymbol{v}\right) - (p - p^\varepsilon, \nabla\cdot\boldsymbol{v}) = 0. \tag{28}$$

By setting $\boldsymbol{v} := \boldsymbol{u} - \boldsymbol{u}^\varepsilon$ in (28), we obtain

$$\begin{aligned}\nu\left(|\nabla\boldsymbol{u}|^{r-2}\nabla\boldsymbol{u}, \nabla(\boldsymbol{u}-\boldsymbol{u}^\varepsilon)\right) &- \nu\left(|\nabla\boldsymbol{u}^\varepsilon|^{r-2}\nabla\boldsymbol{u}^\varepsilon, \nabla(\boldsymbol{u}-\boldsymbol{u}^\varepsilon)\right)\\ &- (p - p^\varepsilon, \nabla\cdot(\boldsymbol{u}-\boldsymbol{u}^\varepsilon)) = 0.\end{aligned} \tag{29}$$

Next, by using the continuity equation (10_2), (29) becomes

$$\begin{aligned}\nu\left(|\nabla\boldsymbol{u}|^{r-2}\nabla\boldsymbol{u}, \nabla(\boldsymbol{u}-\boldsymbol{u}^\varepsilon)\right) &- \nu\left(|\nabla\boldsymbol{u}^\varepsilon|^{r-2}\nabla\boldsymbol{u}^\varepsilon, \nabla(\boldsymbol{u}-\boldsymbol{u}^\varepsilon)\right)\\ &- (p - p^\varepsilon, \nabla\cdot\boldsymbol{u}^\varepsilon) = 0.\end{aligned} \tag{30}$$

By using (11_2), equation (30) reads

$$\nu\left(|\nabla \boldsymbol{u}|^{r-2}\nabla \boldsymbol{u}, \nabla(\boldsymbol{u}-\boldsymbol{u}^{\varepsilon})\right) - \nu\left(|\nabla \boldsymbol{u}^{\varepsilon}|^{r-2}\nabla \boldsymbol{u}^{\varepsilon}, \nabla(\boldsymbol{u}-\boldsymbol{u}^{\varepsilon})\right) + \varepsilon(p^{\varepsilon}, p^{\varepsilon}) = \varepsilon(p, p^{\varepsilon}). \tag{31}$$

By Theorem 3.2, $p^{\varepsilon} \in L^{r}(\Omega)$. Since $r \geq 2$, this implies that $p^{\varepsilon} \in L^{2}(\Omega)$. Next, by using the strong monotonocity of the r-Laplacian (8), the regularity of p^{ε}, the regularity assumption $p \in L^{2}(\Omega)$, and Young's inequality, we have

$$\nu \left\|\boldsymbol{u}-\boldsymbol{u}^{\varepsilon}\right\|_{1,r}^{r} + \varepsilon \left\|p^{\varepsilon}\right\|^{2} \leq \frac{\varepsilon}{2}\left\|p\right\|^{2} + \frac{\varepsilon}{2}\left\|p^{\varepsilon}\right\|^{2}, \tag{32}$$

from which (27) clearly follows. □

5 Error Analysis for the Finite Element Approximation of the Linear Penalty Method

In this section, we turn our attention to the finite element discretization of the linear penalty method (11). First, the existence and uniqueness result as well as the preliminary error bound follow, provided that the approximating subspaces satisfy the discrete LBB condition. Next, we prove that the discrete form of the linear penalty method possesses a *linear* order of convergence via the discrete Sobolev inequalities.

Let $\mathbf{X}_h \subset \mathbf{X} = \mathbf{W}_0^{1,r}(\Omega)$ and $Q \subset Q_h = L_0^{r'}(\Omega)$ be two conforming finite element spaces satisfying the *discrete LBB condition*:

There exists a constant $\beta_h > 0$ such that

$$0 < \beta_h \leq \inf_{q_h \in Q_h} \sup_{\boldsymbol{v} \in \mathbf{X}_h} \frac{(\nabla \cdot \boldsymbol{v}_h, q_h)}{\|q_h\|_{0,r'} \, \|\boldsymbol{v}_h\|_{1,r}}. \tag{33}$$

The *mixed weak formulation* for the finite element discretization of the power-law Stokes problem (10) reads

$$\begin{cases} \nu\left(|\nabla \boldsymbol{u}_h|^{r-2}\, \nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h\right) - (p_h, \nabla \cdot \boldsymbol{v}_h) = (\mathbf{f}, \boldsymbol{v}_h), & \forall\, \boldsymbol{v}_h \in \mathbf{X}_h, \\ (\nabla \cdot \boldsymbol{u}_h, q_h) = 0, & \forall\, q_h \in Q_h, \end{cases} \tag{34}$$

The existence and uniqueness of solutions to (34) was studied in [3,4,6,5].

The *mixed weak formulation* for the finite element discretization of the linear penalty method (11) reads:

Find $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon) \in \mathbf{X}_h \times Q_h$ such that

$$\begin{cases} \nu\,(|\nabla \boldsymbol{u}_h^\varepsilon|^{r-2}\,\nabla \boldsymbol{u}_h^\varepsilon, \nabla \boldsymbol{v}_h) - (p_h^\varepsilon, \nabla\cdot \boldsymbol{v}_h) = (\mathbf{f}, \boldsymbol{v}_h), & \forall\ \boldsymbol{v}_h \in \mathbf{X}_h, \\ (\nabla\cdot \boldsymbol{u}_h^\varepsilon, q_h) + \varepsilon(p_h^\varepsilon, q_h) = 0, & \forall\ q_h \in Q_h. \end{cases} \tag{35}$$

Theorem 5.1 *Let $\mathbf{X}_h \subset \mathbf{X} = \mathbf{W}_0^{1,r}(\Omega)$ and $Q \subset Q_h = L_0^{r'}(\Omega)$ be two conforming finite element spaces satisfying the discrete LBB condition* (33)*. Then, there exists a unique solution $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon) \in \mathbf{X}_h \times Q_h$ to* (35).

Proof. The proof follows along the same lines as the proof of Theorem 3.2. We just sketch it below.

We start by constructing the functional $J_\varepsilon^h : \mathbf{X}_h \to \mathbb{R}$ given by

$$J_\varepsilon^h(\boldsymbol{u}_h) := \frac{\nu}{r}\,\|\nabla \boldsymbol{u}_h\|_{0,r}^r + \frac{1}{2\,\varepsilon}\,\|\nabla\cdot \boldsymbol{u}_h\|^2 - (\mathbf{f}, \boldsymbol{u}_h), \qquad \forall \boldsymbol{u}_h \in \mathbf{X}_h. \tag{36}$$

Next, we prove that J_ε^h is strictly convex, and therefore the minimization problem

$$\min_{\boldsymbol{v}_h \in \mathbf{X}_h} J_\varepsilon^h(\boldsymbol{v}_h). \tag{37}$$

has a unique solution $\boldsymbol{u}_h^\varepsilon \in \mathbf{X}_h$. Then, by using the discrete LBB condition (33), we prove that (37) and (35) are equivalent. Since (37) has a unique solution, we infer that (35) has a unique solution as well, which proves the theorem. □

We now establish the main result which is that (35), the discretization of the linear penalty method, satisfies the error estimate $\|\boldsymbol{u}_h - \boldsymbol{u}_h^\epsilon\|_{1,r} \le C(h)\epsilon$. We start by proving some supporting lemmas.

The first lemma is an *a priori* bound for $\nabla\cdot \boldsymbol{u}_h^\varepsilon$.

Lemma 5.1 *Let $(\boldsymbol{u}_h, p_h)$ solve* (34) *and $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon)$ solve* (35)*. Then*

$$\|\nabla\cdot \boldsymbol{u}_h^\varepsilon\|_r \le C(h)\,\varepsilon, \tag{38}$$

where C is a generic constant depending on h, r, Ω, and $\mathbf{f}$, but not on ε.

Proof. First, by subtracting (35_1) from (34_1), we get

$$\nu\,(|\nabla \boldsymbol{u}_h|^{r-2}\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - \nu\,(|\nabla \boldsymbol{u}_h^\varepsilon|^{r-2}\nabla \boldsymbol{u}_h^\varepsilon, \nabla \boldsymbol{v}_h) \\ - (p_h - p_h^\varepsilon, \nabla\cdot \boldsymbol{v}_h) = 0. \tag{39}$$

By setting $\boldsymbol{v} := \boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon$ in (39), we obtain

$$\nu\left(|\nabla \boldsymbol{u}_h|^{r-2}\nabla \boldsymbol{u}_h, \nabla(\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon)\right) - \nu\left(|\nabla \boldsymbol{u}_h^\varepsilon|^{r-2}\nabla \boldsymbol{u}_h^\varepsilon, \nabla(\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon)\right) \\ - (p_h - p_h^\varepsilon, \nabla\cdot(\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon)) = 0. \tag{40}$$

Next, by using the continuity equation (34_2), (40) becomes

$$\nu\left(|\nabla \boldsymbol{u}_h|^{r-2}\nabla \boldsymbol{u}_h, \nabla(\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon)\right) - \nu\left(|\nabla \boldsymbol{u}_h^\varepsilon|^{r-2}\nabla \boldsymbol{u}_h^\varepsilon, \nabla(\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon)\right) \\ - (p_h - p_h^\varepsilon, \nabla\cdot \boldsymbol{u}_h^\varepsilon) = 0. \tag{41}$$

From (35_2) and (41), we immediately have

$$\nu\left(|\nabla \boldsymbol{u}_h|^{r-2}\nabla \boldsymbol{u}_h, \nabla(\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon)\right) - \nu\left(|\nabla \boldsymbol{u}_h^\varepsilon|^{r-2}\nabla \boldsymbol{u}_h^\varepsilon, \nabla(\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon)\right) \\ + (p_h^\varepsilon + \frac{1}{\varepsilon}\nabla\cdot \boldsymbol{u}_h^\varepsilon, \nabla\cdot \boldsymbol{u}_h^\varepsilon) = 0. \tag{42}$$

Rearranging and using the strong monotonicity of the r-Laplacian (8), we obtain

$$(p_h^\varepsilon + \frac{1}{\varepsilon}\nabla\cdot \boldsymbol{u}_h^\varepsilon, \nabla\cdot \boldsymbol{u}_h^\varepsilon) \le 0. \tag{43}$$

Since $\boldsymbol{u}_h^\varepsilon \in \mathbf{W}^{1,r}(\Omega)$, we have

$$\nabla\cdot \boldsymbol{u}_h^\varepsilon \in L^r(\Omega). \tag{44}$$

Thus, (43) and the Cauchy-Schwartz inequality yield

$$\frac{1}{\varepsilon}\,\|\nabla\cdot \boldsymbol{u}_h^\varepsilon\|_2^2 \le \|p_h^\varepsilon\|_{r'}\,\|\nabla\cdot \boldsymbol{u}_h^\varepsilon\|_r. \tag{45}$$

Now, from the discrete Sobolev inequality, e.g. that found in [15], we have

$$\|\nabla\cdot \boldsymbol{u}_h^\varepsilon\|_r \le C(h)\|\nabla\cdot \boldsymbol{u}_h^\varepsilon\|_2,$$

from which (38) is readily obtained. □

Remark 5.1 *Notice that Lemma 5.1 does not require that $p \in L^2(\Omega)$, as in Theorem 4.1. The only requirement is that $p_h^\varepsilon \in Q = L_0^{r'}(\Omega)$.*

The next lemma bounds the $\mathbf{W}^{1,r}$ norm of $(p_h - p_h^\varepsilon)$ in terms of the $\mathbf{W}^{1,r}$ norm of $(\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon)$.

Lemma 5.2 *Let* $(\boldsymbol{u}_h, p_h)$ *solve* (34) *and* $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon)$ *solve* (35). *Then*

$$\|p_h - p_h^\varepsilon\|_{0,r'} \leq C\, \|\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon\|_{1,r}^{r-1}, \tag{46}$$

where C *is a generic constant depending on* ν, r, β_h, Ω, *and* $\mathbf{f}$, *but not on* ε.

Proof. By letting $q := p_h - p_h^\varepsilon \in L_0^{r'}(\Omega)$ in the discrete LBB condition (33), we get

$$\|p_h - p_h^\varepsilon\|_{r'} \leq C \sup_{\boldsymbol{v}_h \in \mathbf{X}_h} \frac{(p_h - p_h^\varepsilon, \nabla \cdot \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{1,r}}. \tag{47}$$

By using (39), (47) becomes

$$\|p_h - p_h^\varepsilon\|_{r'} \leq C \sup_{\boldsymbol{v}_h \in \mathbf{X}_h} \frac{\nu\left(|\nabla \boldsymbol{u}_h|^{r-2}\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h\right) - \nu\left(|\nabla \boldsymbol{u}_h^\varepsilon|^{r-2}\nabla \boldsymbol{u}_h^\varepsilon, \nabla \boldsymbol{v}_h\right)}{\|\boldsymbol{v}_h\|_{1,r}}. \tag{48}$$

Finally, (46) follows from (48) and the Lipschitz continuity of the r-Laplacian (9). □

The next lemma bounds the $\mathbf{W}^{1,r}$ norm of $(\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon)$ in terms of the L^r norm of $\nabla \cdot \boldsymbol{u}_h^\varepsilon$.

Lemma 5.3 *Let* $(\boldsymbol{u}_h, p_h)$ *solve* (34) *and* $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon)$ *solve* (35). *Then*

$$\|\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon\|_{1,r} \leq C\, \|\nabla \cdot \boldsymbol{u}_h^\varepsilon\|_r, \tag{49}$$

where C *is a generic constant depending on* ν, r, β_h, Ω, *and* $\mathbf{f}$, *but not on* ε.

Proof. Equation (41) implies that

$$\begin{aligned}\nu\left(|\nabla \boldsymbol{u}_h|^{r-2}\nabla \boldsymbol{u}_h, \nabla(\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon)\right) - \nu\left(|\nabla \boldsymbol{u}_h^\varepsilon|^{r-2}\nabla \boldsymbol{u}_h^\varepsilon, \nabla(\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon)\right) \\ \leq \|p_h - p_h^\varepsilon\|_{r'}\, \|\nabla \cdot \boldsymbol{u}_h^\varepsilon\|_r.\end{aligned} \tag{50}$$

By using Lemma 5.2, equation (50) becomes

$$\begin{aligned}\nu\left(|\nabla \boldsymbol{u}_h|^{r-2}\nabla \boldsymbol{u}_h, \nabla(\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon)\right) - \nu\left(|\nabla \boldsymbol{u}_h^\varepsilon|^{r-2}\nabla \boldsymbol{u}_h^\varepsilon, \nabla(\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon)\right) \\ \leq C\, \|\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon\|_{1,r}^{r-1}\, \|\nabla \cdot \boldsymbol{u}_h^\varepsilon\|_r.\end{aligned} \tag{51}$$

Therefore, the result follows from (51) and the strong monotonicity of the r-Laplacian (8). □

The next theorem, which is the main result of this paper, proves error estimates for the convergence of $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon)$ to $(\boldsymbol{u}_h, p_h)$.

Theorem 5.2 *Let Ω be a bounded, connected, Lipshitz continuous domain in $\mathbb{R}^d$. Let $2 \leq r < \infty$ and r' be its conjugate. Let $(\boldsymbol{u}_h, p_h)$ be the unique solution of (34), $\boldsymbol{u}^\varepsilon$ the unique solution of (37), and*

$$p_h^\varepsilon := \frac{1}{\varepsilon\,|\Omega|} \int_\Omega \nabla \cdot \boldsymbol{u}_h^\varepsilon \, d\,\mathbf{x} - \frac{1}{\varepsilon} \nabla \cdot \boldsymbol{u}_h^\varepsilon. \tag{52}$$

Then, there exists a generic constant C, depending on ν, r, Ω, β_h and $\mathbf{f}$, but not on ε, such that

$$\|\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon\|_{1,r} \leq C\,\varepsilon \qquad \textit{and} \tag{53}$$

$$\|p_h - p_h^\varepsilon\|_{0,r'} \leq C\,\varepsilon^{r-1} \tag{54}$$

Proof. The estimate (53) clearly follows from Lemmas 5.1 and 5.3. Next, the estimate (54) clearly follows from Lemma 5.2 and (53). □

6 Numerical Experiments

In this section, we present numerical experiments which investigate the rate of convergence of $\boldsymbol{u}_h^\varepsilon$ to $\boldsymbol{u}_h$ *with respect to the penalty parameter* ε. Here $\boldsymbol{u}_h$ and $\boldsymbol{u}_h^\varepsilon$ are the finite element approximations of the power-law Stokes problem (34) and linear penalty method applied to the power-law Stokes problem (35), respectively.

In our numerical experiments, we will treat $\boldsymbol{u}$, the solution to the power-law Stokes problem (10), as an unknown, since our main objective is to investigate the rate of convergence of $\boldsymbol{u}_h^\varepsilon$ to $\boldsymbol{u}_h$, and not to $\boldsymbol{u}$. Thus, we investigate the convergence rate for

$$\|\boldsymbol{u}_h^\varepsilon - \boldsymbol{u}_h^{2\varepsilon}\|_{1,r} \leq \|\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon\|_{1,r} + \|\boldsymbol{u}_h - \boldsymbol{u}_h^{2\varepsilon}\|_{1,r}.$$

As model problem, we consider the power-law Stokes problem with homogeneous Dirichlet boundary conditions

$$-\nabla \cdot \Big[(|\nabla \boldsymbol{u}|^{r-2})\nabla \boldsymbol{u}\Big] + \nabla p = \mathbf{f}, \quad \text{in } \Omega \tag{55}$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad \text{in } \Omega \tag{56}$$

$$\boldsymbol{u} = 0, \quad \text{on } \partial\Omega, \tag{57}$$

ϵ	$\|\boldsymbol{u}_h^\epsilon - \boldsymbol{u}_h^{2\epsilon}\|_{1,r}$ ($r=3$)	cvge. rate	$\|\boldsymbol{u}_h^\epsilon - \boldsymbol{u}_h^{2\epsilon}\|_{1,r}$ ($r=7/2$)	cvge. rate	$\|\boldsymbol{u}_h^\epsilon - \boldsymbol{u}_h^{2\epsilon}\|_{1,r}$ ($r=4$)	cvge. rate
$10^{-4}/4$	$2.252099 \cdot 10^{-3}$		$5.698729 \cdot 10^{-3}$		$1.122629 \cdot 10^{-2}$	
$10^{-4}/8$	$1.517193 \cdot 10^{-3}$	0.57	$4.232039 \cdot 10^{-3}$	0.43	$8.871262 \cdot 10^{-3}$	0.34
$10^{-4}/16$	$9.917802 \cdot 10^{-4}$	0.61	$3.148196 \cdot 10^{-3}$	0.43	$7.030412 \cdot 10^{-3}$	0.34
$10^{-4}/32$	$6.219753 \cdot 10^{-4}$	0.67	$2.335926 \cdot 10^{-3}$	0.43	$5.567619 \cdot 10^{-3}$	0.34
$10^{-4}/64$	$3.710372 \cdot 10^{-4}$	0.75	$1.689801 \cdot 10^{-3}$	0.47	$4.393043 \cdot 10^{-3}$	0.34
predicted		0.50		0.40		0.33

Table 1
Numerical results for the nonlinear penalty method (58), $r > 2$.

where the computational domain $\Omega(0,1) \times (0,1) \subset \mathbb{R}^2$ and $\boldsymbol{f} = [f_1, 0]^t$, with

$$f_1(x,y) = \begin{cases} sin(2\pi x), \text{ if } y > 1/2, x < 1/2, \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

We discretize the resulting system (55), (58) using the Taylor-Hood finite element pair and use a mesh of characteristic length $h = 1/32$. For the nonlinearities in our models, we use a Newton iteration.

Nonlinear Penalty Function Method. For the nonlinear penalty function method, equation (56) is replaced by the nonlinear relationship

$$\epsilon\, p^\epsilon + (|\nabla \cdot \boldsymbol{u}^\epsilon|^{r-2}) \nabla \cdot \boldsymbol{u}^\epsilon = 0. \tag{58}$$

In [16], it was proved that the expected convergence rate for this method is $O(\epsilon^{1/(r-1)})$. Table 1 presents numerical results for the convergence rate of $\|\boldsymbol{u}_h^\varepsilon - \boldsymbol{u}_h^{2\varepsilon}\|_{1,r}$ as $\varepsilon \to 0$, for $r > 2$ ($r = 3, r = 7/2$ and $r = 4$). It is clear that the numerical results follow the theoretical convergence rate of $O(\varepsilon^{\frac{1}{r-1}})$ proved in [16].

Linear Penalty Function Method. For the linear penalty function method, equation (56) is replaced by the linear relationship

$$\epsilon\, p^\epsilon + \nabla \cdot \boldsymbol{u}^\epsilon = 0. \tag{59}$$

The predicted convergence rate for this method is $O(\epsilon)$, as proved in Theorem 5.2 (equation (53)). Tables 2 and 3 present numerical results for the convergence rate of $\|\boldsymbol{u}_h^\varepsilon - \boldsymbol{u}_h^{2\varepsilon}\|_{1,r}$ as $\varepsilon \to 0$, for $r > 2$ ($r = 3, r = 7/2$ and

ϵ	$\|\boldsymbol{u}_h^\epsilon - \boldsymbol{u}_h^{2\epsilon}\|_{1,r}$ ($r = 3$)	cvge. rate	$\|\boldsymbol{u}_h^\epsilon - \boldsymbol{u}_h^{2\epsilon}\|_{1,r}$ ($r = 7/2$)	cvge. rate	$\|\boldsymbol{u}_h^\epsilon - \boldsymbol{u}_h^{2\epsilon}\|_{1,r}$ ($r = 4$)	cvge. rate
$10^{-4}/4$	$3.252774 \cdot 10^{-5}$		$3.607812 \cdot 10^{-5}$		$4.125926 \cdot 10^{-5}$	
$10^{-4}/8$	$1.626424 \cdot 10^{-5}$	1.00	$1.803949 \cdot 10^{-5}$	1.00	$2.063023 \cdot 10^{-5}$	1.00
$10^{-4}/16$	$8.132209 \cdot 10^{-6}$	1.00	$9.019856 \cdot 10^{-6}$	1.00	$1.031527 \cdot 10^{-5}$	1.00
$10^{-4}/32$	$4.066128 \cdot 10^{-6}$	1.00	$4.509955 \cdot 10^{-6}$	1.00	$5.15767 \cdot 10^{-6}$	1.00
$10^{-4}/64$	$2.033070 \cdot 10^{-6}$	1.00	$2.254984 \cdot 10^{-6}$	1.00	$2.578846 \cdot 10^{-6}$	1.00
predicted		1.00		1.00		1.00

Table 2
Numerical results for the linear penalty method (59), $r > 2$.

ϵ	$\|\boldsymbol{u}_h^\epsilon - \boldsymbol{u}_h^{2\epsilon}\|_{1,r}$ ($r = 3/2$)	cvge. rate	$\|\boldsymbol{u}_h^\epsilon - \boldsymbol{u}_h^{2\epsilon}\|_{1,r}$ ($r = 7/4$)	cvge. rate
$10^{-4}/4$	$2.745215 \cdot 10^{-5}$		$2.731418 \cdot 10^{-5}$	
$10^{-4}/8$	$1.372631 \cdot 10^{-5}$	1.00	$1.365725 \cdot 10^{-5}$	1.00
$10^{-4}/16$	$6.863214 \cdot 10^{-6}$	1.00	$6.828667 \cdot 10^{-6}$	1.00
$10^{-4}/32$	$3.431622 \cdot 10^{-6}$	1.00	$3.414344 \cdot 10^{-6}$	1.00
$10^{-4}/64$	$1.715815 \cdot 10^{-6}$	1.00	$1.707175 \cdot 10^{-6}$	1.00
predicted		1.00		1.00

Table 3
Numerical results for the linear penalty method (59), $r < 2$.

$r = 4$), and $r > 2$ ($r = 3/2$ and $r = 7/4$), respectively. In both cases, the expected (linear) order of convergence is verified.

7 Conclusions

In this paper, we have analyzed both mathematically and numerically, an improvement in the penalty method for the power-law Stokes problem. In particular, we proved that the finite element discretization of a *linear* penalty method for the power-law Stokes problem yields higher order accuracy than the finite element discretization of the known nonlinear penalty method. The theoretical error estimate were supported by numerical experiments. We also proved the existence and uniqueness of solutions to the continuous and finite element discretization of the linear penalty method applied to the power-law Stokes problem.

The extension of the improved linear penalty method to the Navier-Stokes equations will be the subject of a future study. This could have a significant impact on turbulent flow computations, were computational efficiency is paramount.

References

[1] R.A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[2] C. Amrouche and V. Girault. Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension. *Czech. Math. J.*, 44:109–140, 1994.

[3] J. Baranger, P. Georget, and K. Najib. Error estimates for a mixed finite element method for for a non-Newtonian flow. *J. of non-Newtonian Fluid Mech.*, 23:415–421, 1987.

[4] J. Baranger and K. Najib. Analyse numérique des écoulements quasi-Newtoniens dont la viscosité obéit à la loi puissance ou la loi de Carreau. *Numer. Math.*, 58:34–49, 1990.

[5] J. W. Barrett and W. B. Liu. Finite element error analysis of a quasi-Newtonian flow obeying the Carreau or power law. *Numer. Math.*, 64(4):433–453, 1993.

[6] J.W. Barrett and W. B. Liu. Finite element approximation of the p-Laplacian. *Math. Comp.*, 61:523–537, 1993.

[7] L. C. Berselli, T. Iliescu, and W. J. Layton. *Mathematics of large eddy simulation of turbulent flows.* Scientific Computation. Springer-Verlag, Berlin, 2006.

[8] D. V. Boger, A. Cabelli, and A. L. Halmos. The behavior of a power-law fluid flowing through a sudden expansion. *A. I. Ch. E. Journal*, 21:540–549, 1975.

[9] S.S. Chow. Finite element error estimates for non-linear elliptic equations of monotone type. *Numer. Math.*, 54:373–393, 1989.

[10] C. Cuvelier, A. Segal, and A. A. van Steenhoven. *Finite element methods and Navier-Stokes equations.* D. Reidel Publishing Co., Boston, 1986.

[11] P. England and J. Jackson. Active deformation of the continents. *Ann. Rev. Earth Planet. Sci.*, 17:197–226, 1989.

[12] V. Girault and P.A. Raviart. *Finite Element Approximation of the Navier-Stokes Equations.* Springer-Verlag, New York, 1979.

[13] M.D. Gunzburger. *Finite element methods for viscous incompressible flows.* Academic Press, Boston, 1989.

[14] T. J. R. Hughes, W. K. Liu, and A. Brooks. Finite element analysis of incompressible viscous flows by the penalty function formulation. *J. Comput. Phys.*, 30(1):1–60, 1979.

[15] W.J. Layton. A nonlinear, subgridscale model for incompressible viscous flow problems. *SIAM J. Sci. Comput.*, 17(2):347–357, 1996.

[16] L. Lefton and D. Wei. A penalty method for approximations of the stationary power-law Stokes problem. *Elec. J. Diff. Eq.*, 2001(7):1–12, 2001.

[17] L. Lefton and D. Wei. Penalty finite element approximations of the stationary power-law Stokes problem. *J. Numer. Math.*, 11:301–322, 2003.

[18] J.L. Lions. *Quelques Méthodes de Résolutios des Problémes aux Limites Non Linéaires.* Dunod, Paris, 1969.

[19] T. J. Liu, S. Wen, and J. Tsou. Three-dimensional finite element analysis of polymeric fluid flow in an extrusion die. Part I: Entrance effects. *Polymer Eng. Sci.*, 34(10):827–834, 1994.

[20] G. Minty. Monotone (nonlinear) operators in Hilbert space. *Duke Math J.*, 29:341–346, 1961.

[21] A. Quarteroni and A. Valli. *Numerical approximation of partial differential equations.* Springer-Verlag, Berlin, 1994.

[22] J. N. Reddy and V. A. Padhye. A penalty finite element model for axisymmetric flows of non-Newtonian fluids. *Numer. Methods Partial Differential Equations*, 4(1):33–56, 1988.

[23] P. Sagaut. *Large eddy simulation for incompressible flows.* Scientific Computation. Springer-Verlag, Berlin, third edition, 2006.

[24] J. Shen. On error estimates of the penalty method for unsteady Navier-Stokes equations. *SIAM J. Numer. Anal.*, 32(2):386–403, 1995.

[25] J.S. Smagorinsky. General circulation experiments with the primitive equations. *Mon. Weather Review*, 91:99–164, 1963.

[26] L. J. Sonder and P. C. England. Vertical averages of rheology of the continental litosphere. *Earth Planet. Sci. Lett.*, 77:81–90, 1989.

[27] R. Temam. Une méthode d'approximation de la solution des équations de Navier-Stokes. *Bull. Soc. Math. France*, 96:115–152, 1968.

[28] D. Wei. Penalty approximations to the stationary power-law Navier-Stokes problem. *Numer. Funct. Anal. Optim.*, 22:749–765, 2001.