Global regularity for the 2D Boussinesq equations with partial viscosity terms

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Abstract

In this paper, we prove the global in time regularity for the 2D Boussinesq system with either the zero diffusivity or the zero viscosity. We also prove that as diffusivity (viscosity) tends to zero, the solutions of the fully viscous equations converge strongly to those of zero diffusion (viscosity) equations. Our result for the zero diffusion system, in particular, solves the Problem no. 3 posed by Moffatt in [R.L. Ricca, (Ed.), Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001, pp. 3–10].

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1. Introduction

The 2D Boussineq system for the incompressible fluid flows in $\mathbb{R}^2$ is

\[
(B) \begin{cases}
\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\nabla p + v \Delta v + \theta e_2, \\
\frac{\partial \theta}{\partial t} + (v \cdot \nabla) \theta = \kappa \Delta \theta, \\
\text{div} \, v = 0, \\
v(x, 0) = v_0(x), \quad \theta(x, 0) = \theta_0(x),
\end{cases}
\]

where $v = (v_1, v_2)$, $v_j = v_j(x, t)$, $j = 1, 2$, $(x, t) \in \mathbb{R}^2 \times (0, \infty)$, is the velocity vector field, $p = p(x, t)$ is the scalar pressure, $\theta(x, t)$ is the scalar temperature, $\nu \geq 0$ is the viscosity, and $\kappa \geq 0$ is the thermal diffusivity, and $e_2 = (0, 1)$. The Boussinesq system has important roles in the atmospheric sciences (see e.g. [9]).

The global in time regularity of $(B)$ with $\nu > 0$ and $\kappa > 0$ is well-known (see e.g. [2]). On the other hand, the regularity/singularity questions of the case of $(B)$ with $\kappa = \nu = 0$ is an outstanding open problem in the mathematical fluid mechanics (see e.g. [3,4,6,11] for studies in this direction). Even the regularity problem for 'partial viscosity cases' (i.e. either the zero diffusivity case, $\kappa = 0$ and $\nu > 0$, or the zero viscosity case, $\kappa > 0$ and $\nu = 0$), has been open to the author’s knowledge. Actually, the author has been recently informed of the article by Moffatt, where the question of singularity in the case $\kappa = 0, \nu > 0$ and its possible development in the limit $\kappa \to 0$ is listed as one of the 21st century problems (see the Problem no. 3 in [10]). For this problem very recent progress has been made by Cordoba et al. [5], where the authors proved that special type of singularities, called 'squirt singularities', are absent. In this paper, we consider both partial viscosity cases, and prove the global in time regularity for both of the cases. We also prove that as diffusivity (viscosity) tends to zero the solutions of $(B)$ converge strongly to those of zero diffusivity(viscosity) equations. In particular, the Problem no. 3 in [10] is solved. More precise statements of our results are stated in Theorems 1.1 and 1.2 below. After submission of this paper the author was informed of the preprint version of [8], where the authors obtained results similar to the part of Theorem 1.1.

For later references we write down the zero diffusivity Boussinesq equations:

\[
(B_1) \begin{cases}
\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\nabla p + v \Delta v + \theta e_2, \\
\frac{\partial \theta}{\partial t} + (v \cdot \nabla) \theta = 0, \\
\text{div} \, v = 0, \\
v(x, 0) = v_0(x), \quad \theta(x, 0) = \theta_0(x),
\end{cases}
\]

where $\nu > 0$ is fixed. For this system the following is our main result.
Theorem 1.1. Let $v > 0$ be fixed, and $\text{div } v_0 = 0$. Let $m > 2$ be an integer, and $(v_0, \theta_0) \in H^m(\mathbb{R}^2)$. Then, there exists a unique local classical solution $(v, \theta)$ with $\theta \in C([0, \infty); H^m(\mathbb{R}^2))$ and $v \in C([0, \infty); H^m(\mathbb{R}^2)) \cap L^2(0, T; H^{m+1}(\mathbb{R}^2))$ of the system $(B_1)$. Moreover, for each $s < m$, the solutions $(v, \theta)$ of $(B)$ converge to the corresponding solutions of $(B_1)$ in $C([0, T]; H^s(\mathbb{R}^2))$ as $\kappa \to 0$.

We also write down the zero viscosity Boussinesq equations

\begin{equation}
(B_2) \begin{cases}
\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p + \theta e_2, \\
\frac{\partial \theta}{\partial t} + (v \cdot \nabla)\theta = \kappa \Delta \theta, \\
\text{div } v = 0, \\
v(x, 0) = v_0(x), \quad \theta(x, 0) = \theta_0(x),
\end{cases}
\end{equation}

where $\kappa > 0$ is given. The following is our result on $(B_2)$.

Theorem 1.2. Let $\kappa > 0$ be fixed, and $\text{div } v_0 = 0$. Let $m > 2$ be an integer. Let $m > 2$ be an integer, and $(v_0, \theta_0) \in H^m(\mathbb{R}^2)$. Then, there exists unique solutions $(v, \theta)$ with $v \in C([0, \infty); H^m(\mathbb{R}^2))$ and $\theta \in C([0, \infty); H^m(\mathbb{R}^2)) \cap L^2(0, T; H^{m+1}(\mathbb{R}^2))$ of the system $(B_2)$. Moreover, for each $s < m$, the solutions $(v, \theta)$ of $(B)$ converge to the corresponding solutions of $(B_2)$ in $C([0, T]; H^s(\mathbb{R}^2))$ as $v \to 0$.

2. The proof of Theorem 1.1

We first recall the following result on the system $(B)$ with $\kappa = v = 0$, proved in [3,6] (see also [4]):

Theorem 2.1. Suppose $(v_0, \theta_0) \in H^m(\mathbb{R}^2)$ with $m > 2$ being an integer. Then, there exists a unique local classical solution $(v, \theta) \in C([0, T_1]; H^m(\mathbb{R}^2))$ for some $T_1 = T_1(\|v_0\|_{H^m}, \|\theta_0\|_{H^m})$. Moreover, the solution remains in $H^m(\mathbb{R}^2)$ up to time $T > T_1$, namely $(v, \theta) \in C([0, T]; H^m(\mathbb{R}^2))$ if and only if

\begin{equation}
\int_0^T \|\nabla \theta(t)\|_{L^\infty} dt < \infty. \tag{2.1}
\end{equation}

By obvious changes of the proof in [3] we can easily infer that a similar conclusion holds for system $(B_1)$ and $(B_2)$, respectively. Hence, for the proof of the global regularity part of Theorems 1.1 and 1.2, it suffices to prove estimate (2.1) for all $T \in (0, \infty)$ for the classical solutions $(v, \theta)$ of $(B_1)$ and $(B_2)$.
2.1. Preliminary estimates

Let $T > 0$ be a given fixed time. From the second equation of $(B_1)$ we immediately have

$$
\| \theta(t) \|_{L^p} \leq \| \theta_0 \|_{L^p} \quad \forall t \in [0, T], \quad p \in [1, \infty].
$$

(2.2)

Taking $L^2$ inner product the first equation of $(B_1)$ with $v$, we have, after integration by part

$$
\frac{1}{2} \frac{d}{dt} \| v \|^2_{L^2} + v \| \nabla v \|^2_{L^2} \leq \| \theta \|_{L^2} \| v \|_{L^2}.
$$

Hence,

$$
\frac{1}{2} \frac{d}{dt} \| v \|^2_{L^2} \leq \| \theta \|_{L^2} \| v \|_{L^2} \leq \| \theta_0 \|_{L^2} \| v \|_{L^2},
$$

where we used (2.2) for $p = 2$. Hence, $\frac{d}{dt} \| v \|_{L^2} \leq \| \theta_0 \|$, and we obtain

$$
\| v(t) \|_{L^2} \leq \| v_0 \|_{L^2} + \| \theta_0 \|_{L^2} T \quad \forall t \in [0, T].
$$

(2.3)

Taking the operation curl on both sides of the first equation of (B), we obtain

$$
\omega_t + (v \cdot \nabla) \omega = -\partial_{x_1} + v \Delta \omega,
$$

(2.4)

where $\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1$. Let $p \geq 2$. Multiplying (2.4) by $\omega |\omega|^{p-2}$ and integrating it over $\mathbb{R}^2$, we find, after integration by part

$$
\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^2} |\omega|^p \, dx + (p - 1) v \int_{\mathbb{R}^2} |\nabla \omega|^2 |\omega|^{p-2} \, dx
$$

$$
= \frac{1}{p} \int_{\mathbb{R}^2} (v \cdot \nabla) |\omega|^p \, dx - \int_{\mathbb{R}^2} \partial_{x_1} \omega |\omega|^{p-2} \, dx
$$

$$
= -\frac{1}{p} \int_{\mathbb{R}^2} \text{div} v |\omega|^p \, dx + (p - 1) \int_{\mathbb{R}^2} \theta_0 \omega_{x_1} |\omega|^{p-2} \, dx
$$

$$
\leq \frac{(p - 1) v}{2} \int_{\mathbb{R}^2} |\nabla \omega|^2 |\omega|^{p-2} \, dx + \frac{(p - 1)}{2v} \int_{\mathbb{R}^2} \theta^2 |\omega|^{p-2} \, dx
$$

$$
\leq \frac{(p - 1) v}{2} \int_{\mathbb{R}^2} |\nabla \omega|^2 |\omega|^{p-2} \, dx + \frac{(p - 1)}{2v} \| \theta \|_{L^p}^2 \| \omega \|_{L^p}^{p-2}.
$$
Carrying over the term \( \frac{(p-1)v}{2} \int_{\mathbb{R}^2} |\nabla \omega|^2 |\omega|^{p-2} \, dx \) to the left-hand side, we find

\[
\frac{1}{p} \frac{d}{dt} \| \omega \|_{L^p}^p + \frac{(p-1)v}{2} \int_{\mathbb{R}^2} |\nabla \omega|^2 |\omega|^{p-2} \, dx \leq \frac{(p-1)}{2v} \| \theta \|_{L^p}^2 \| \omega \|_{L^p}^{p-2}.
\] (2.5)

For \( p = 2 \), in particular, after integration over \([0, T]\) we obtain

\[
\| \omega(t) \|_{L^2}^2 + v \int_0^T \| \nabla \omega(s) \|_{L^2}^2 \, ds \leq 2 \| \omega_0 \|_{L^2}^2 + \frac{2}{v} \| \theta_0 \|_{L^2}^2 T \quad \forall t \in [0, T].
\] (2.6)

Hence, we find that, by Hölder’s inequality,

\[
\int_0^T \| \nabla \omega(s) \|_{L^2}^2 \, ds \leq C \sqrt{T} \left( \int_0^T \| \nabla \omega(s) \|_{L^2}^2 \, ds \right)^{1/2}
\]

\[
\leq C \| \omega_0 \|_{L^2} \sqrt{T} + C \| \theta_0 \|_{L^2} T \quad \forall t \in [0, T].
\] (2.7)

On the other hand, from (2.5), we have for \( p \in [2, \infty) \)

\[
\| \omega(t) \|_{L^p}^2 \leq \| \omega_0 \|_{L^p}^2 + \frac{(p-1)}{v} \| \theta_0 \|_{L^p}^2 T \leq \left( \| \omega_0 \|_{L^p} + \frac{\sqrt{p-1}}{\sqrt{v}} \| \theta_0 \|_{L^p} \sqrt{T} \right)^2
\]

and

\[
\| \omega(t) \|_{L^p} \leq \| \omega_0 \|_{L^p} + \frac{\sqrt{p-1}}{\sqrt{v}} \| \theta_0 \|_{L^p} \sqrt{T} \quad \forall t \in [0, T], \ p \in [2, \infty).
\] (2.8)

We now recall the Gagliardo–Nirenberg interpolation inequality in \( \mathbb{R}^2 \).

\[
\| f \|_{L^\infty} \leq C \| f \|_{L^p}^{\frac{p-2}{p-2}} \| Df \|_{L^p}^{\frac{p}{p-2}}, \quad f \in W^{1,p}(\mathbb{R}^2), \ p > 2.
\] (2.9)

By this and the Calderon–Zygmund inequality combined with estimates (2.3) and (2.8) for \( p \in (2, \infty) \) we find

\[
\| v(t) \|_{L^\infty} \leq C \| v(t) \|_{L^2}^{\frac{p-2}{p-2}} \| \nabla v(t) \|_{L^p}^{\frac{p}{p-2}} \| \omega(t) \|_{L^p}^{\frac{p-2}{p-2}} \| v(t) \|_{L^2}^{\frac{p}{p-2}} \| \omega(t) \|_{L^p}^{\frac{p}{p-2}}
\]

\[
\leq C(v_0, \theta_0, T, v, p) \quad \forall t \in [0, T].
\] (2.10)
2.2. $W^{2,p}$ estimate for $v$

We take the derivative operation $D = (\partial_x, \partial_y)$ on (2.4), and then take $L^2$ inner product with $D\omega|D\omega|^{p-2}$, $p > 2$. After integration by part we obtain

$$
\frac{1}{p} \frac{d}{dt} \|D\omega\|_{L^p}^p + (p-1)v \int_{\mathbb{R}^2} |D^2\omega|^2 |D\omega|^{p-2} \, dx
$$

$$
= - \int_{\mathbb{R}^2} [D(v \cdot \nabla)\omega] D\omega|D\omega|^{p-2} \, dx - \int_{\mathbb{R}^2} D\theta_x D\omega|D\omega|^{p-2} \, dx
$$

$$
= (p-1) \int_{\mathbb{R}^2} [(v \cdot \nabla)\omega] D^2\omega|D\omega|^{p-2} \, dx + (p-1) \int_{\mathbb{R}^2} \theta_x D^2\omega|D\omega|^{p-2} \, dx
$$

$$
\leq \frac{(p-1)v}{4} \int_{\mathbb{R}^2} |D^2\omega|^2 |D\omega|^{p-2} \, dx + \frac{(p-1)}{v} \int_{\mathbb{R}^2} |v(x)|^2 |D\omega|^{p} \, dx
$$

$$
+ \frac{(p-1)v}{4} \int_{\mathbb{R}^2} |D^2\omega|^2 |D\omega|^{p-2} \, dx + \frac{(p-1)}{v} \int_{\mathbb{R}^2} |\nabla\theta|^2 |D\omega|^{p-2} \, dx,
$$

where we used the inequality, $ab \leq \frac{a^2}{4} + \frac{b^2}{v}$. Carrying over the first and the third terms to the left-hand side, we have

$$
\frac{1}{p} \frac{d}{dt} \|D\omega\|_{L^p}^p + \frac{(p-1)v}{2} \int_{\mathbb{R}^2} |D^2\omega|^2 |D\omega|^{p-2} \, dx
$$

$$
\leq \frac{(p-1)}{v} \int_{\mathbb{R}^2} |v(x)|^2 |D\omega|^{p} \, dx + \frac{(p-1)}{v} \int_{\mathbb{R}^2} |\nabla\theta|^2 |D\omega|^{p-2} \, dx
$$

$$
\leq \frac{(p-1)}{v} \|v\|_{L^\infty} \|D\omega\|_{L^p}^p + \frac{2(p-1)}{pv} \|\nabla\theta\|_{L^p}^p + \frac{(p-1)(p-2)}{pv} \|D\omega\|_{L^p}^p,
$$

where we used Young’s inequality, $a^2 b^{p-2} \leq \frac{2}{p} a^p + \frac{p-2}{p} b^p$ for $p \geq 2$. Recalling the estimate of $\|v(t)\|_{L^\infty}$ in (2.10), we find that

$$
\frac{d}{dt} \|D\omega\|_{L^p}^p \leq C \|D\omega\|_{L^p}^p + C \|\nabla\theta\|_{L^p}^p \quad \forall t \in [0, T],
$$

(2.11)

where $C = C(v_0, \theta_0, T, v, p)$.

Now taking $\nabla^\perp = (-\partial_x, \partial_y)$ to the second equation of (B1), we obtain

$$
\nabla^\perp \theta_t + (v \cdot \nabla) \nabla^\perp \theta = \nabla^\perp \theta \cdot \nabla v.
$$

(2.12)
Taking $L^2$ inner product (2.12) with $\nabla^\perp \theta |\nabla^\perp \theta|^{p-2}$, we deduce, after integration by part, that

$$
\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla \theta|^p \, dx = \frac{1}{p} \int_{\mathbb{R}^2} (v \cdot \nabla)|\nabla \theta|^p \, dx + \int_{\mathbb{R}^2} (\nabla^\perp \theta \cdot \nabla) v \cdot \nabla^\perp \theta |\nabla \theta|^{p-2} \, dx
\leq \int_{\mathbb{R}^2} |\nabla v||\nabla \theta|^p \, dx.
$$

Hence, for $p > 2$ we have

$$
\frac{d}{dt} \|\nabla \theta\|^p_{L^p} \leq \frac{1}{p} \int_{\mathbb{R}^2} (v \cdot \nabla)|\nabla \theta|^p \, dx + \int_{\mathbb{R}^2} (\nabla^\perp \theta \cdot \nabla) v \cdot \nabla^\perp \theta |\nabla \theta|^{p-2} \, dx
\leq \int_{\mathbb{R}^2} |\nabla v||\nabla \theta|^p \, dx.
$$

where $C = (v_0, \theta_0, T, v, p)$, and we used the following form of the Brezis–Wainger inequality [1] (see also [7])

$$
\|f\|_{L^\infty} \leq C(1 + \|\nabla f\|_{L^2} \|D^2 v\|_{L^2}) \left[ 1 + \log^+ (\|D^2 v\|_{L^p}) \right] \|\nabla \theta\|^p_{L^p}
\leq C(1 + \|\omega\|_{L^2} \|D\omega\|_{L^2}) \left[ 1 + \log^+ (\|D\omega\|_{L^p} + \|\nabla \theta\|^p_{L^p}) \right] \|\nabla \theta\|^p_{L^p},
$$

(2.13)

for $f \in L^2(\mathbb{R}^2) \cap W^{1,p}(\mathbb{R}^2)$, which holds for $p > 2$, and the Calderon–Zygmund inequality as well as estimate (2.6). Adding (2.11) and (2.13) together, and setting $X(t) = \|\nabla \theta\|^p_{L^p} + \|D\omega\|^p_{L^p}$, we find that

$$
\frac{dX}{dt} \leq C(1 + \|D\omega(t)\|_{L^2} \|D\omega\|_{L^2}) \left( 1 + \log^+ X \right) X \leq C(1 + \|D\omega(t)\|_{L^2} \|D\omega\|_{L^2}) \left( 1 + \log^+ X \right) X
$$

for all $t \in [0, T]$, where $C = C(v_0, \theta_0, T, v, p)$. By Gronwall’s lemma we have

$$
X(t) \leq X(0) \exp \left\{ CT + C \int_0^T \|D\omega(s)\|_{L^2} \, ds \right\} \quad \forall t \in [0, T],
$$

which, combined with estimate (2.7), implies that for $p > 2$

$$
\|D\omega(t)\|_{L^p} \leq C(v_0, \theta_0, T, v, p) \quad \forall t \in [0, T].
$$

(2.15)
By the Gagliardo–Nirenberg (2.9) and the Calderon–Zygmund inequalities we have
\[
\| \nabla v(t) \|_{L^\infty} \leq C \| \nabla v(t) \|_p^{\frac{p}{2p-2}} \| D^2 v(t) \|_L^p \| \omega(t) \|_L^\frac{p-2}{2} \| D \omega(t) \|_L^{\frac{p}{2}} \\
\leq C(v_0, \theta_0, T, v, p) \quad \forall t \in [0, T], \quad p \in (2, \infty],
\]
(2.16)
where we used the Gagliardo–Nirenberg inequality (2.9), estimates (2.8) and (2.15).
From the first part of the inequalities of (2.13), we find that
\[
\frac{d}{dt} \| \nabla \theta_0 \|_{L^p} \leq \| \nabla v \|_{L^\infty} \| \nabla \theta_0 \|_{L^p}
\]
and by Gronwall’s lemma
\[
\| \nabla \theta(t) \|_{L^p} \leq \| \nabla \theta_0 \|_{L^p} \exp \left( \int_0^t \| \nabla v(s) \|_{L^\infty} \, ds \right).
\]
(2.17)
Given \( \varepsilon, R > 0 \), let us define a set \( A_{\varepsilon, R}(t) \) by
\[
A_{\varepsilon, R}(t) = \{ x \in \mathbb{R}^2 \mid | \nabla \theta(x,t) | > \| \nabla \theta(t) \|_{L^\infty} - \varepsilon, \ |x| < R \}.
\]
Then, applying the following \( L^p \) interpolation inequality to (2.17):
\[
\| f \|_{L^p} \leq \| f \|_{L^2}^{\frac{2}{p}} \| f \|_{L^\infty}^{1-\frac{2}{p}}, \quad 2 \leq p \leq \infty,
\]
we obtain
\[
(\| \nabla \theta(t) \|_{L^\infty} - \varepsilon) |A_{\varepsilon, R}(t)| \frac{1}{p} \| \nabla \theta_0 \|_{L^2}^{\frac{2}{p}} \| \nabla \theta_0 \|_{L^\infty}^{1-\frac{2}{p}} \exp \left( \int_0^t \| \nabla v(s) \|_{L^\infty} \, ds \right),
\]
(2.18)
where \( |A_{\varepsilon, R}(t)| \) is the Lebesgue measure of the set \( A_{\varepsilon, R}(t) \), which is finite. Passing first \( p \to \infty \), and then \( \varepsilon \to 0 \) in (2.18), we have
\[
\| \nabla \theta(t) \|_{L^\infty} \leq \| \nabla \theta_0 \|_{L^\infty} \exp \left( \int_0^T \| \nabla v(s) \|_{L^\infty} \, ds \right)
\]
\[
\leq C \quad \forall t \in [0, T],
\]
(2.19)
where \( C = C(\| v_0 \|_{W^{2,p}}, \| \theta_0 \|_{W^{2,p}}, T, p, v) \), and we used estimate (2.16). Since we have the embedding, \( H^m(\mathbb{R}^2) \hookrightarrow W^{2,p}(\mathbb{R}^2) \), for all \( m > 2 \) and \( p > 2 \) we attained estimate (2.1) for any given \( T \in (0, \infty) \) and for all \( v_0, \theta_0 \in H^m(\mathbb{R}^2) \) with \( m > 2 \).

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2.3. Vanishing diffusivity limit

Let \((v, p, \theta)\) and \((\tilde{v}, \tilde{p}, \tilde{\theta})\) be solutions of \((B_1)\) and \((B)\), respectively, with the same initial conditions \((v_0, \theta_0)\). We first observe that all the estimates derived in (i) and (ii) above are also valid for solutions of \((B)\). Moreover, these estimates are independent of \(\kappa\). Summarizing these estimates, we have the key \(\kappa\)-independent estimates for the solutions \((\tilde{v}, \tilde{\theta})\)

\[
\|\nabla \tilde{v}\|_{L^\infty} + \|\nabla \tilde{\theta}\|_{L^\infty} + \|\tilde{v}\|_{W^{2,p}} + \|\tilde{\theta}\|_{W^{2,p}} \leq C(\|v_0\|_{W^{2,p}}, \|\theta_0\|_{W^{2,p}}, v, T, p). \tag{2.20}
\]

From \((B)\)–\((B)_1\) for \(\Theta = \theta - \tilde{\theta}, \ P = p - \tilde{p}, \ V = v - \tilde{v}\) we obtain

\[
\Theta_t + (v \cdot \nabla)\Theta + (V \cdot \nabla)\tilde{\theta} = \kappa \Delta \Theta + \kappa \Delta \tilde{\theta} \tag{2.21}
\]

and

\[
V_t + (v \cdot \nabla)V + (V \cdot \nabla)\tilde{v} = -\nabla P + \Theta e_2 + \nu \Delta V \tag{2.22}
\]

together with \(\text{div} \ V = 0\). Taking \(L^2\) inner product (2.21) with \(\Theta\), after integration by part we have

\[
\frac{1}{2} \frac{d}{dt} \|\Theta\|^2_{L^2} + \kappa \|\nabla \Theta\|^2_{L^2} = -\int_{\mathbb{R}^2} (V \cdot \nabla)\tilde{\theta} \Theta \, dx - \kappa \int_{\mathbb{R}^2} \nabla \tilde{\theta} \cdot \nabla \Theta \, dx \leq \|\nabla \tilde{\theta}\|_{L^\infty} \|V\|_{L^2} \|\Theta\|_{L^2} + \kappa \|\nabla \tilde{\theta}\|_{L^2} \|\nabla \Theta\|_{L^2} \\
\leq C \|V\|_{L^2}^2 + C \|\Theta\|_{L^2}^2 + \frac{\kappa}{2} \|\nabla \tilde{\theta}\|_{L^2}^2 + \frac{\kappa}{2} \|\nabla \Theta\|_{L^2}^2,
\]

where \(C = C(v_0, \theta_0, T, v)\), and we have used estimate (2.20). Carrying over the term, \(\frac{\kappa}{2} \|\nabla \Theta\|_{L^2}^2\) to the left-hand side, and using estimate (2.20), we obtain that

\[
\frac{d}{dt} \|\Theta\|^2_{L^2} + \kappa \|\nabla \Theta\|^2_{L^2} \leq C \|\Theta\|_{L^2}^2 + C \|V\|_{L^2}^2 + C \kappa \|\nabla \tilde{\theta}\|_{L^2}^2. \tag{2.23}
\]

On the other hand, we take \(L^2\) inner product (2.22) with \(V\), and integrate by part to obtain

\[
\frac{1}{2} \frac{d}{dt} \|V\|^2_{L^2} + \nu \|\nabla V\|^2_{L^2} = -\int_{\mathbb{R}^2} (V \cdot \nabla)\tilde{v} \cdot V \, dx + \int_{\mathbb{R}^2} \Theta e_2 V \, dx \leq \|\nabla \tilde{v}\|_{L^\infty} \|V\|_{L^2}^2 + \|\Theta\|_{L^2} \|\tilde{v}\|_{L^2} \\
\leq C(\|V\|_{L^2}^2 + \|\Theta\|_{L^2}^2), \tag{2.24}
\]
where $C = C(v_0, \theta_0, T, v)$, where we used (2.20) again. Adding (2.24) to (2.23), and setting $X(t) = \|\Theta(t)\|_{L^2}^2 + \|V(t)\|_{L^2}^2$, we obtain that

$$\frac{d}{dt}X(t) \leq CX(t) + C \|\nabla\tilde{\theta}\|_{L^2}^2.$$ 

By Gronwall’s lemma we find that

$$X(t) \leq X(0)e^{CT} + C \int_0^T \|\nabla\tilde{\theta}(s)\|_{L^2}^2 e^{C(t-s)} ds$$

where we used the fact that $X(0) = 0$ and estimate (2.20). Hence, we obtain

$$\sup_{0 \leq t \leq T} (\|v(t) - \tilde{v}(t)\|_{L^2} + \|\theta(t) - \tilde{\theta}(t)\|_{L^2}) \leq C \sqrt{\kappa},$$

(2.25)

where $C = C(v_0, \theta_0, T, v)$. From the Gagliardo–Nirenberg interpolation inequality, and estimate (2.20) together with the embedding, $H^m(\mathbb{R}^2) \hookrightarrow W^{2,p}(\mathbb{R}^2)$ for $m > 2$, we deduce that for $0 \leq s < m$

$$\sup_{0 \leq t \leq T} \|v(t) - \tilde{v}(t)\|_{H^s} \leq C \sup_{0 \leq t \leq T} \|v(t) - \tilde{v}(t)\|_{H^m}^{1-\sigma} \sup_{0 \leq t \leq T} \|v(t) - \tilde{v}(t)\|_{L^2}^{\sigma}$$

$$\leq C \kappa \frac{m-s}{m}, \text{ where } \sigma = 1 - \frac{s}{m} \text{ and } C = C(v_0, \theta_0, T, v, s, m)$$

and similarly for $\|\theta - \tilde{\theta}\|_{H^s}$, and we obtain the desired convergence $(\tilde{v}, \tilde{\theta}) \rightarrow (v, \theta)$ in $C([0, T]; H^s((\mathbb{R}^2)))$ as $\kappa \rightarrow 0$.

Remark after the proof: The local existence and finite time blow-up criterion for solutions in the functional setting, $H^m(\mathbb{R}^2), m > 2$, which is proved in ([3,6]) can be easily modified using function spaces $W^{2,p}(\mathbb{R}^2), p > 2$. Combining this with the above proof, we can actually prove the following:

**Corollary 2.1.** Let $2 < p < \infty$, and $(v_0, \theta_0) \in W^{2,p}(\mathbb{R}^2)$. Then, there exists a unique solution $(v, \theta) \in C([0, \infty); W^{2,p}(\mathbb{R}^2))$ of the system $(B_1)$. Moreover, for each $q \in [1, p)$ and $T \in (0, \infty)$, the solutions $(v, \theta)$ of $(B)$ converge to the corresponding solutions of $(B_1)$ in $C([0, T]; W^{1,q}(\mathbb{R}^2))$ as $\kappa \rightarrow 0$. 

3. The proof of Theorem 1.2

Similar to the preliminary remark in the beginning of the previous section, in order to prove the global regularity part of Theorem 1.2 we have only to prove estimate (2.1) for the classical solution of \((B_2)\) for all \(T \in (0, \infty)\).

3.1. Preliminary estimates

Taking \(L^2\) inner product the second equation of \((B_2)\) with \(\theta\), we have immediately

\[
\frac{1}{2} \frac{d}{dt} \|\theta\|^2_{L^2} + \kappa \|\nabla \theta\|^2_{L^2} = 0.
\]

Integrating this over \([0, T]\) we have

\[
\frac{1}{2} \|\theta(t)\|^2_{L^2} + \int_0^T \|\nabla \theta\|^2_{L^2} dt \leq \frac{1}{2} \|\theta_0\|^2_{L^2} \quad \forall t \in [0, T]. \tag{3.1}
\]

Next, taking \(L^2\) inner product of the first equation of \((B_2)\) with \(v\), after integration by part we have

\[
\frac{1}{2} \frac{d}{dt} \|v\|^2_{L^2} = \int_{\mathbb{R}^2} \theta e_2 \cdot v \, dx \leq \|\theta\|_{L^2} \|v\|_{L^2}.
\]

Combining this with (3.1), we easily obtain

\[
\|v(t)\|_{L^2} \leq \|v_0\|_{L^2} + \int_0^T \|\theta(s)\|_{L^2} ds = \|v_0\|_{L^2} + T \|\theta_0\|_{L^2} \tag{3.2}
\]

for all \(t \in [0, T]\). Taking the curl of the first equation of \((B_2)\), we have

\[
\omega_t + (v \cdot \nabla) \omega = -\theta x_1. \tag{3.3}
\]

Taking \(L^2\) inner product (3.3) with \(\omega\), and integrating by part, we deduce

\[
\frac{1}{2} \frac{d}{dt} \|\omega\|^2_{L^2} \leq \int_{\mathbb{R}^2} |\nabla \theta| |\omega| \, dx \leq \|\nabla \theta\|_{L^2} \|\omega\|_{L^2}
\]

and

\[
\frac{d}{dt} \|\omega\|_{L^2} \leq \|\nabla \theta\|_{L^2}.
\]
Hence, using estimate (3.1), we derive
\[ \| \omega(t) \|_{L^2} \leq \int_0^T \| \nabla \theta \|_{L^2} \, dt + \| \omega_0 \|_{L^2} \]
\[ \leq T^{1/2} \left( \int_0^T \| \nabla \theta \|_{L^2}^2 \, dt \right)^{1/2} + \| \omega_0 \|_{L^2} \]
\[ \leq \frac{T^{1/2}}{\sqrt{2}} \| \theta_0 \|_{L^2} + \| \omega_0 \|_{L^2} \quad \forall t \in [0, T]. \quad (3.4) \]

3.2. \( W^{1,p} \) estimate for \((\theta, v)\)

Using operation \( \nabla \perp \) on the second equation of \((B_2)\), we have
\[ \nabla \perp \theta + (v \cdot \nabla) \nabla \perp \theta = (\nabla \perp \theta \cdot \nabla)v + \kappa \Delta \nabla \perp \theta. \quad (3.5) \]

We now take scalar product (3.5) in \( L^2 \) by \( \nabla \perp \theta \| \nabla \perp \theta \| p^{-2} \), \( p > 2 \); after integration by part we obtain
\[ \frac{1}{p} \frac{d}{dt} \| \nabla \perp \theta \|_{L^p}^p + (p - 1) \kappa \int_{\mathbb{R}^2} |D^2 \theta|^2 |\nabla \perp \theta|^{p-2} \, dx \]
\[ = \int_{\mathbb{R}^2} (\nabla \perp \theta \nabla)v \cdot \nabla \perp \theta |\nabla \perp \theta|^{p-2} \, dx \]
\[ = -(p - 1) \int_{\mathbb{R}^2} v \cdot (\nabla \perp \theta \cdot \nabla) \nabla \perp \theta |\nabla \perp \theta|^{p-2} \, dx \]
\[ \leq (p - 1) \int_{\mathbb{R}^2} |v| |\nabla \perp \theta| |D^2 \theta||\nabla \perp \theta|^{p-2} \, dx \]
\[ \leq \frac{(p - 1)}{2\kappa} \int_{\mathbb{R}^2} |v|^2 |\nabla \perp \theta|^p \, dx + \frac{(p - 1)\kappa}{2} \int_{\mathbb{R}^2} |D^2 \theta|^2 |\nabla \perp \theta|^{p-2} \, dx, \]

where we used the inequality, \( ab \leq \frac{a^2}{2\kappa} + \frac{kb^2}{2} \). We carry over the second term to the left-hand side to have
\[ \frac{d}{dt} \| \nabla \theta \|_{L^p}^p + \frac{p(p - 1)\kappa}{2} \int_{\mathbb{R}^2} |D^2 \theta|^2 |\nabla \theta|^{p-2} \, dx \]
\[ \leq \frac{(p - 1)p}{2\kappa} \| v \|_{L^\infty}^2 \| \nabla \theta \|_{L^p}^p \]
\[ \leq C \left( 1 + \| v \|_{L^2} + \| \nabla v \|_{L^2} \right)^2 \left( 1 + \log^+ \left( \| \nabla v \|_{L^p}^p \right) \right) \| \nabla \theta \|_{L^p}^p \]
\[ \leq C(1 + \|v\|_{L^2}^2 + \|\omega\|_{L^2}^2) \left[ 1 + \log^+(\|\omega\|_{L^p}^p + \|\nabla \theta\|_{L^p}^p) \right] \|\nabla \theta\|_{L^p}^p \]
\[ \leq C \left[ 1 + \log^+(\|\omega\|_{L^p}^p + \|\nabla \theta\|_{L^p}^p) \right] \|\nabla \theta\|_{L^p}^p, \tag{3.6} \]

where we applied the Brezis–Wainger inequality (2.14) retaining the power, \( \frac{1}{2} \), of the log-term preserved, the Calderon–Zygmund inequality, and estimates (3.2) and (3.4). On the other hand, taking \( L^2 \) inner product (3.3) with \( \|D^2\|_{L^p}^p \), we obtain

\[ \frac{1}{p} \frac{d}{dt} \|\omega\|_{L^p}^p + \frac{1}{p} \int_{\mathbb{R}^2} (v \cdot \nabla) |\omega|^p dx = - \int_{\mathbb{R}^2} \theta \nabla^2 \omega |\omega|^{p-2} dx \]
\[ \leq \int_{\mathbb{R}^2} \|\nabla \theta\| |\omega|^{p-1} dx \]
\[ \leq \frac{1}{p} \|\nabla \theta\|_{L^p}^p + \frac{(p-1)}{p} \|\omega\|_{L^p}^p, \tag{3.7} \]

where we used Young’s inequality, \( ab^{p-1} \leq \frac{1}{p} a^p + \frac{p-1}{p} b^p \), \( 1 < p < \infty \). Adding (3.7) to (3.6), and setting \( X(t) = \|\nabla \theta(t)\|_{L^p}^p + \|\omega\|_{L^p}^p \), we have

\[ \frac{d}{dt} X(t) \leq C (1 + \log X(t)) X(t) \ \forall t \in [0, T]. \]

The Gronwall lemma provides us with

\[ X(t) \leq X(0) e^{CT} \ \forall t \in [0, T]. \]

Hence,

\[ \|\nabla \theta(t)\|_{L^p}^p + \|\omega\|_{L^p}^p \leq C(v_0, \theta_0, T, p, \kappa). \tag{3.8} \]

We also note that similar to (2.10), estimate (3.8), combined with (3.2) and (2.9) implies that

\[ \|v(t)\|_{L^\infty} \leq C(v_0, \theta_0, T, p) \ \forall t \in [0, T]. \tag{3.9} \]

### 3.3. \( W^{2,p} \) estimate for \( \theta \)

Taking operation \( D^2 \) on the second equation of \((B_2)\), and then taking \( L^2 \) inner product of this with \( D^2 \theta \), \( D^2 \theta \), \( \omega \), \( \kappa \), \( p > 2 \), we have after integration by part

\[ \frac{1}{p} \frac{d}{dt} \|D^2 \theta\|_{L^p}^p + (p-1) \kappa \int_{\mathbb{R}^2} |D^2 \theta|^2 |D^3 \theta|^{p-2} dx \]
\[ = - \int_{\mathbb{R}^2} D^2 (v \cdot \nabla) \theta \, D^2 \theta \, |D^2 \theta|^{p-2} \, dx = (p-1) \int_{\mathbb{R}^2} D[(v \cdot \nabla) \theta] \, D^2 \theta \, |D^2 \theta|^{p-2} \, dx \]

\[ = (p-1) \int_{\mathbb{R}^2} Dv \cdot D\theta \, D^2 \theta \, |D^2 \theta|^{p-2} \, dx + (p-1) \int_{\mathbb{R}^2} [(v \cdot \nabla) \theta] \, D^2 \theta \, |D^2 \theta|^{p-2} \, dx \]

\[ \leq \left( \frac{p-1}{\kappa} \right) \|\nabla \theta\|_{L^\infty}^2 \int_{\mathbb{R}^2} |\nabla v|^2 \, |D^2 \theta|^{p-2} \, dx + \left( \frac{p-1}{4} \right) \int_{\mathbb{R}^2} |D^3 \theta|^2 \, |D^2 \theta|^{p-2} \, dx \]

\[ + \left( \frac{p-1}{\kappa} \right) \|v\|_{L^p}^2 \int_{\mathbb{R}^2} |D^2 \theta|^p \, dx + \left( \frac{p-1}{4} \right) \int_{\mathbb{R}^2} |D^3 \theta|^2 \, |D^2 \theta|^{p-2} \, dx, \]

where we used the inequality, \( ab \leq \frac{a^2}{\kappa} + \frac{k^2}{4}, \) again. Carrying over the terms, \( \left( \frac{p-1}{4} \right) \int_{\mathbb{R}^2} |D^3 \theta|^2 \, |D^2 \theta|^{p-2} \, dx \) to the left-hand side, we derive

\[
\frac{d}{dt} \|D^2 \theta\|_{L^p}^p \leq C \|\nabla \theta\|_{L^\infty}^2 \|\nabla v\|_{L^p}^2 \|D^2 \theta\|_{L^p}^{p-2} + C \|v\|_{L^\infty}^2 \|D^2 \theta\|_{L^p}^p \\
\leq C \|\nabla \theta\|_{L^p}^{2p-4} \|\omega\|_{L^p}^2 \|D^2 \theta\|_{L^p}^{p-2} + C \|v\|_{L^\infty}^2 \|D^2 \theta\|_{L^p}^p \\
\leq C + C \|D^2 \theta\|_{L^p}^p,
\]

where we used the Gagliardo–Nirenberg interpolation inequality (2.9), estimates (3.8), (3.9), and Young’s inequality (note \( p - \frac{2p-4}{2p-2} < p \) when \( p > 2 \)). Thanks to Gronwall’s lemma, we have the estimate

\[ \|D^2 \theta(t)\|_{L^p} \leq C(v_0, \theta_0, T, p, \kappa) \quad \forall t \in [0, T], \quad \forall p > 2. \]

Using the interpolation inequality (2.9) as previously, we obtain that

\[ \|\nabla \theta(t)\|_{L^\infty} \leq C \quad \forall t \in [0, T], \quad (3.10) \]

where \( C = C(\|v_0\|_{W^2,p}, \|\theta_0\|_{W^2,p}, p, \kappa). \) Similar to the proof of Theorem 1.1, we have the embedding, \( H^m(\mathbb{R}^2) \hookrightarrow W^2,p(\mathbb{R}^2), \) for all \( m > 2 \) and \( p > 2, \) and thus we attained estimate (2.1) for all \( T \in (0, \infty) \) and for all \( v_0, \theta_0 \in H^m(\mathbb{R}^2) \) with \( m > 2. \)

### 3.4. Vanishing viscosity limit

Let \((v, \theta)\) and \((\tilde{v}, \tilde{\theta})\) be solutions of \((B_2)\) and \((B)\), respectively with the same initial conditions \((v_0, \theta_0)\). Similar to the case of the zero diffusivity problem in Section 2, we first note that all the estimates in (i)–(iii) above are valid for solutions of \((B)\) also, and these estimates are independent of \(v\). The key \(v\)-independent estimate for the solutions \((\tilde{v}, \tilde{\theta})\) is

\[ \|\nabla \tilde{v}\|_{L^\infty} + \|\nabla \tilde{\theta}\|_{L^\infty} + \|\tilde{v}\|_{W^2,p} + \|\tilde{\theta}\|_{W^2,p} \leq C(\|v_0\|_{W^2,p}, \|\theta_0\|_{W^2,p}, \kappa, T, p). \quad (3.11) \]
From (B)–(B\_2) we obtain for \( \Theta = \theta - \tilde{\theta} \), \( P = p - \tilde{p} \), \( V = v - \tilde{v} \)

\[ \Theta_t + (v \cdot \nabla)\Theta + (V \cdot \nabla)\tilde{\theta} = \kappa\Delta\Theta \]  
(3.12)

with \( \text{div} \ V = 0 \), and

\[ V_t + (v \cdot \nabla)V + (V \cdot \nabla)\tilde{v} = -\nabla P + \Theta e_2 + v\Delta V + v\Delta\tilde{v}. \]  
(3.13)

Taking \( L^2 \) inner product (3.12) with \( \Theta \), after integration by part we have

\[ \frac{1}{2} \frac{d}{dt} \| \Theta \|^2_{L^2} + \kappa \| \nabla \Theta \|^2_{L^2} = -\int_{\mathbb{R}^2} (V \cdot \nabla)\tilde{\theta} \Theta \, dx \]
\[ \leq \| \nabla \tilde{\theta} \|_{L^\infty} \| V \|_{L^2} \| \Theta \|_{L^2} \]
\[ \leq C \| V \|^2_{L^2} + C \| \Theta \|^2_{L^2}, \]

where \( C = C(v_0, \theta_0, T, \kappa) \), and we have used estimate (3.11). Hence,

\[ \frac{d}{dt} \| \Theta \|^2_{L^2} \leq C \| \Theta \|^2_{L^2} + C \| V \|^2_{L^2}. \]  
(3.14)

On the other hand, we take \( L^2 \) inner product (3.13) with \( V \), and integrate by part to obtain:

\[ \frac{1}{2} \frac{d}{dt} \| V \|^2_{L^2} + v \| \nabla V \|^2_{L^2} = -\int_{\mathbb{R}^2} (V \cdot \nabla)\tilde{v} \cdot V \, dx + \int_{\mathbb{R}^2} \Theta e_2 \cdot V \, dx - v \int_{\mathbb{R}^2} \nabla \tilde{v} \cdot \nabla V \, dx \]
\[ \leq \| \nabla \tilde{v} \|_{L^\infty} \| V \|^2_{L^2} + \| \Theta \|_{L^2} \| V \|_{L^2} + \| \nabla \tilde{v} \|_{L^2} \| \nabla V \|_{L^2} \]
\[ \leq C (\| V \|^2_{L^2} + \| \Theta \|^2_{L^2}) + \frac{v}{2} \| \nabla V \|^2_{L^2} + \frac{v}{2} \| \nabla \tilde{v} \|^2_{L^2}, \]

where \( C = C(v_0, \theta_0, T, \kappa) \), and used estimate (3.11) again. Carrying over the term, \( \frac{v}{2} \| \nabla V \|^2_{L^2} \), to the left-hand side, we obtain that

\[ \frac{d}{dt} \| V \|^2_{L^2} \leq C (\| V \|^2_{L^2} + \| \Theta \|^2_{L^2}) + \frac{v}{2} \| \nabla \tilde{v} \|^2_{L^2}. \]  
(3.15)

Adding (3.15) to (3.14), and setting \( X(t) = \| \Theta(t) \|^2_{L^2} + \| V(t) \|^2_{L^2} \), we obtain that

\[ \frac{d}{dt} X(t) \leq CX(t) + C v \| \nabla \tilde{v} \|^2_{L^2}. \]
By Gronwall’s lemma we find that

\[ X(t) \leq X(0)e^{Ct} + CV \int_0^t \| \nabla \tilde{v}(s) \|_{L^2}^2 e^{C(t-s)} \, ds \]

\[ \leq Ce^{CT} \int_0^T \| \nabla \tilde{v}(t) \|_{L^2}^2 \, dt \leq CV, \]

where we used (3.11) and the fact that \( X(0) = 0 \). Hence, we obtain

\[
\sup_{0 \leq t \leq T} (\| v(t) - \tilde{v}(t) \|_{L^2} + \| \theta(t) - \tilde{\theta}(t) \|_{L^2}) \leq C\sqrt{v}, \tag{3.16}
\]

where \( C = C(v_0, \theta_0, T, \kappa) \). Similar to the case of the vanishing diffusivity limit, the interpolation inequality, and the uniform in \( v \) estimate (3.11) led us to the estimate, for \( 0 \leq s < m \),

\[
\sup_{0 \leq t \leq T} \| v(t) - \tilde{v}(t) \|_{H^s} \leq C \sup_{0 \leq t \leq T} \| v(t) - \tilde{v}(t) \|_{L^2}^\sigma \| v(t) - \tilde{v}(t) \|_{H^m}^{1-\sigma},
\]

\[
\leq C(\| v_0 \|_{H^m} + \| \tilde{v}_0 \|_{H^m})^{1-\sigma} \sup_{0 \leq t \leq T} \| v(t) - \tilde{v}(t) \|_{L^2}^\sigma
\]

\[
\leq CV^{\frac{m-s}{2m}}, \quad \text{where } \sigma = 1 - \frac{s}{m} \text{ and } C = C(v_0, \theta_0, T, \kappa, s, m)
\]

and similarly for \( \| \theta - \tilde{\theta} \|_{H^s} \), we obtain the desired convergence \((\tilde{v}, \tilde{\theta}) \to (v, \theta)\) in \( C([0, T]; H^s(\mathbb{R}^2)) \) as \( v \to 0 \).

**Remark after the proof:** Similar to the remark at the end of Section 2 we can actually prove the following:

**Corollary 3.1.** Let \( 2 < p < \infty \), and \( (v_0, \theta_0) \in W^{2,p}(\mathbb{R}^2) \). Then, there exist unique solutions \((v, \theta) \in C([0, \infty); W^{2,p}(\mathbb{R}^2))\) of the system \((B_2)\). Moreover, for each \( q \in [1, p) \), \( T \in (0, \infty) \), the solutions \((v, \theta)\) of \((B)\) converge to the corresponding solutions of \((B_1)\) in \( C([0, T]; W^{1,q}(\mathbb{R}^2)) \) as \( v \to 0 \).

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