Number and Operations

Summer 2005

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Course Description

Numbers and Operations:

The course will enable teachers to:

1. Understand and be able to explain the mathematics that underlies the procedures used for operating on whole numbers and rational numbers.

2. Understand and be able to explain the distinctions among whole numbers, integers, rational numbers, and real numbers, and that field axioms hold or do not hold depending on the types of numbers being used.

3. Convert easily among fractions, decimals, and percents.

4. Demonstrate facility in using number and operation properties, including mental computation and computational estimation.

5. Understand and be able to explain fundamental ideas of number theory as they apply to middle school mathematics.

6. Make sense of large and small numbers and use scientific notation.

7. Apply proportions appropriately and provide explanations.

* The activities and explorations throughout this document were compiled by Susan Hagen (hagen@vt.edu) and Stephanie Behm (sbehm@vt.edu). Special thanks to Gwen Lloyd for providing numerous activities and resources in the development of this course.

Texts and Supplemental Readings

Recommended Texts

Barnett, C., Goldenstein, D., & Jackson, B. (1994). *Fraction, decimals, ratios, & percents: Hard to teach and hard to learn?* Portsmouth, NH: Heinemann. (ISBN: 0-435-08357-0)

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (2004). Bits and Pieces II (Student Addition). *Connected Mathematics Project.* Prentice Hall. (ISBN: 0131808125)

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (2004). Prime Time (Student Edition). *Connected Mathematics Project.* Prentice Hall. (ISBN: 0131808079)

Reflections on Number (Student Edition). *Mathematics in Context.* Holt, Rinehart and Winston. (ISBN: 0030717043)

(Purchase one copy for each school)
National Council of Teachers of Mathematics. (1998). *The teaching and learning of algorithms in school mathematics*. Reston, VA: National Council of Teachers of Mathematics. (ISBN: 0-87353-440-9)

Additional References

Bassarear, T. (1997). *Mathematics for elementary school teachers: Explorations*. New York: Houghton Mifflin Company.

Beckmann, S. (2003). *Mathematics for elementary teachers: Volume 1- Numbers and operations.* Boston: Addison Wesley.

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Driscoll, M. (1999). *Fostering algebraic thinking: A guide for teachers grades 6–10.* Portsmouth, NH: Heinemann.

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Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding It Up: Helping Children Learn Mathematics*. Available online at www.nap.edu/books/0309069955/html/.

Knuth, E., & Peressini, D. (2001). Unpacking the nature of discourse in mathematics classrooms. *Mathematics Teaching in the Middle School, 6*(5), 320-325.

Liljedahl, P., & Zazkis, R. (2004). Understanding Primes: The Role of Representation. *Journal for Research in Mathematics Education, 35*(3), 164-186.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education, 30*(2), 122-147.

Musser, Gary L., Burger, William F., & Peterson, Blake E. (2003). *Mathematics for Elementary Teachers*. New York: John Wiley & Sons, Inc.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Randolph, T. D., & Sherman, H. J. (2001). Alternative algorithms: Increasing options, reducing errors. *Teaching Children Mathematics, 7*(8), 480-484.

Thinking Rationally About Fractions, Decimals and Percents (www.pen.k12.va.us/VDOE/Instruction/ Math/FractionsDecimalsPercent.pdf).

Wu, Z. (2001). Multiplying Fractions. *Teaching Children Mathematics, 8*(3), 174-177.

Yang, D., & Reys, R. (2001). One fraction, many solution paths. *Mathematics Teaching in the Middle School, 7*(3), 164-166.

Recommended Readings for Instructors

Barnett, C., Goldenstein, D., & Jackson, B. (1994). *Fraction, decimals, ratios, & percents: Hard to teach and hard to learn? Facilitator's Discussion Guide.* Portsmouth, NH: Heinemann. (ISBN: 0-435-08358-9)

National Council of Teachers of Mathematics. (1994). *Curriculum and evaluation standards for school mathematics: Understanding rational numbers and proportions*. Reston, VA: National Council of Teachers of Mathematics. (ISBN: 0-87353-325-9)

Sowder, J. T., Philipp, R. A., Armstrong, B. E., & Schappelle, B. P. (1998). *Middle grade teachers' mathematical knowledge and its relationship to instruction: A research monograph*. NY: State University of New York. (ISBN: 0-7914-3842-2)

Operating with Whole Numbers

Content Covered:

- Sets
- Associative and commutative properties
- Alternative methods of addition and subtraction
- Alternative methods of multiplication and division
- Division by zero

SOLs and NCTM Standards:

Associated SOLs

3.8 The student will solve problems involving the sum or difference of two whole numbers, each 9,999 or less, with or without regrouping, using various computational methods, including calculators, paper and pencil, mental computation, and estimation.

3.9 The student will represent multiplication and division, using area and set models, and create and solve problems that involve multiplication of two whole numbers, one factor 99 or less and the second factor 5 or less.

4.5 The student will estimate whole-number sums and differences and describe the method of estimation. Students will refine estimates, using terms such as *closer to, between*, and *a little more than*.

4.6 The student will add and subtract whole numbers written in vertical and horizontal form, choosing appropriately between paper and pencil methods and calculators.

4.7 The student will find the product of two whole numbers when one factor has two digits or fewer and the other factor has three digits or fewer, using estimation and paper and pencil. For larger products (a two-digit numeral times a three-digit numeral), estimation and calculators will be used.

4.8 The student will estimate and find the quotient of two whole numbers, given a one-digit divisor.

5.3 The student will create and solve problems involving addition, subtraction, multiplication, and division of whole numbers, using paper and pencil, estimation, mental computation, and calculators.

5.4 The student, given a dividend of four digits or fewer and a divisor of two digits or fewer, will find the quotient and remainder.

6.7 The student will use estimation strategies to solve multistep practical problems involving whole numbers, decimals, and fractions (rational numbers).

7.3 The student will identify and apply the following properties of operations with real numbers:

a) the commutative and associative properties for addition and multiplication;

b) the distributive property;

c) the additive and multiplicative identity properties;

d) the additive and multiplicative inverse properties; and

e) the multiplicative property of zero.

7.5 The student will formulate rules for and solve practical problems involving basic operations (addition, subtraction, multiplication, and division) with integers.

Associated NCTM Standards

Grades 3 – 5

- understand various meanings of multiplication and division
- understand the effects of multiplying and dividing whole numbers
- develop fluency with basic number combinations for multiplication and division and use these combinations to mentally compute related problems, such as 30x60.
- develop fluency in adding, subtracting, multiplying, and dividing whole numbers
- develop and use strategies to estimate the results of whole-number computations and to judge the reasonableness of such results

Grades 6 – 8

- use the associative and commutative properties of addition and multiplication and the distributive property of multiplication over addition to simplify computations with integers, fractions, and decimals

Problems and Activities (by Content):

Sets

First, a few definitions:

- **Definition of Sets:** A set must be well defined; i.e., for any given object, it must be unambiguous whether or not the object is an element of the set. For example, if a set contains all the chairs in a designated room, then any chair can be determined either to be in or not in the set. If there were no chairs in the room, the set would be called the empty, or null, set, i.e., one containing no elements. A set is usually designated by a capital letter. If A is the set of even numbers between 1 and 9, then $A=\{2, 4, 6, 8\}$. The braces, {}, are commonly used to enclose the listed elements of a set. The elements of a set may be described without actually being listed. If B is the set of real numbers that are solutions of the equation $x^2=9$, then the set can be written as $B=\{ x:x^2=9\}$ or $B=\{ x|x^2=9\}$, both of which are read: B is the set of all x such that $x2=9$; hence B is the set $\{3,-3\}$. (http://www.factmonster.com)

- **Property of Closure:** If we take two *real numbers* and multiply them together, we get another real number. (The real numbers are all the rational numbers and all the irrational numbers.) Because this is always true, we say that the real numbers are "closed under the operation of multiplication": there is no way to escape the set. When you combine any two elements of the set, the result is also included in the set…. You can picture closure as a cage at the zoo. If a cage holds giraffes and the gate remains closed, then the cage will hold only giraffes. If the gate is open, then an elephant or two might be in with the giraffes. Mathematically speaking: always start by knowing what set of numbers you are working with. Pick two or more of those numbers, perform an operation on them, and see if your result is in that set. If it is, then that set is closed under that operation. (http://mathforum.org/dr.math)

1. Suppose that S is a set of whole numbers closed under addition. S contains 3, 27 and 72. List six other elements in S. Why must 24 be in S?

2. A given set contains the number 5. What other numbers must also be in the set if it is closed under addition?

3. Sandy says, “I think of closure as a bunch of numbers locked in a room, and the operation, like addition comes along and links two of the numbers together. As long as the answer is inside the room, the set is closed, but if they have to go outside the room to get a new number for the answer, the set is not closed. Like if you have to open the door, then the set is not closed.” How would your respond to Sandy’s analogy?

4. We know that 8 – 3 = 5. This example seems to indicate that the whole numbers are closed under subtraction. But it is also possible to give an example that indicates that the whole numbers are not closed under subtraction. Demonstrate such an example and explain why your example “wins” over the first example.

5. Explain, in your own words, what it means for a set to be closed under addition.

Associative and commutative properties

6. Addition can be simplified using the associative property of addition. For example:
   26 + 57 = 26 + (4 + 53) = (26 + 4) + 53 = 30 + 53 = 83
   Find different ways to apply the associative property of addition to simplify:
   a. 39 + 68
   b. 25 + 56
   c. 47 + 23

7. Discuss how a 6-year-old would find the answer to ”What is 2 + 7?” If the same child was then asked “What is 7 + 2?” how would the child find the answer to that question? Is there a difference? Explain.

8. Use what you found in #7 above, to find different ways to apply the “commutative” properties of addition and the associative to the problems in #6. How many different ways can you find to work these problems?

Alternative methods of addition and subtraction

9. One of your students says, “When I added 27 and 36, I made the 27 a 30, then I added the 30 and the 36 and got 66 and then subtracted the 3 from the beginning and got 63.” Another student says, “But when I added 27 and 36, I added the 20 and the 30 and got 50, then I knew 6 plus 6 was 12 so 6 plus 7 was 13, and then I added 50 and 13 and got 63.” Can you follow the student’s reasoning here? How would you describe some of their techniques?

10. The partial differences method of subtraction is shown below. This method involves subtracting left to right one column at a time. Study the example and determine the rules applied.

    Problem: 746 – 263 = ?

	7 4 6	
-	2 6 3	
+	5 0 0	Subtract the 100s
-	2 0	Subtract the 10s
+	3	Subtract the 1s
	4 8 3	

    So 746 – 263 = 483.
    Will this method work all the time?

11. Is it helpful to you to see more than one method of doing arithmetic problems? Explain. Do you think it is helpful for all students to see more than one method for doing arithmetic problems? What are your reasons?

12. Using the chart below, determine the different strategies necessary to complete the 100 empty spaces. Are there 100 different strategies?

+	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

Technology:

13. Take some time to think about positive and negative numbers. Work through the following applets from http://matti.usu.edu/nlvm/nav/topic_t_1.html

    - Circle 0
    - Circle 21
    - Circle 99
    - Color Chips – Addition
    - Color Chips – Subtraction

    After working through these applets, talk in your groups about how you could help students develop an understanding of positive and negative numbers through the use of these applets. What ideas do these applets give you about introducing negative numbers in your own classrooms?

Alternative methods of multiplication and division

14. Multiplication and Whole Numbers

    a. Do Reflections on Number Section C "Investigating Algorithms" #1-6 (p. 19-22). Write up your responses to #4, 5.
    b. The Standard Multiplication Algorithm:
    Using the problem below, discuss why each step of the standard multiplication algorithm works. If you'd like, use the questions below as a guide.

		5	6
	x	3	7
	3	9	2
1	6	8	
2	0	7	2

    i. We begin by multiplying 7 and 6. Why do we start with these two digits?
    ii. Instead of placing the product 42 below the 7 and 6, we place only the 2 below them. Why?
    iii. Why do we find the product of 5 and 7 next?
    iv. We add the 4 from the previous step to the 35 (from 5 times 7) and we place the 39 to the left of the 2. Why do we add 4 to 35? What does the 392 represent?
    v. Next we multiply 3 and 6. How do we know to multiply these two digits?

vi. Why don't we place the 8 from this multiplication under the 2 instead of the 9? (Why do we move over?) What does the 168 represent?
vii. Finally we add the two rows to get 2072. If this is multiplication, why are we adding?

c. Do *Reflections on Number* Section C "Investigating Algorithms" #8, 9 (p. 23). Write up your responses.
d. Do *Reflections on Number* Section C "Investigating Algorithms" #10-14 (p. 24-27) and Summary Question #26 (p. 32). Write up your response to #11.

15. A student tells you that multiplying by 5 is a lot like dividing by 2. For example, 48 x 5 = 240, but it is easier just to go 48 divided by 2 = 24 and then affix a zero at the end. Will this method always work? Explain.

16. Another student says dividing by 5 is the same as multiplying by 2 and "dropping" a zero. Can you figure out what this student is saying? Does this work? Explain.

17. Division and Remainders

a. Complete the following problems. Describe a mathematical discussion that might evolve out of doing these problems in your own classroom.
   - Harding Avenue Elementary School is going to the Duck Pond for a picnic. There are 369 students going, and each bus can hold 24 students. How many school buses are needed?
   - Make up a story problem involving 31 divided by 4 in which the answer is 7.
   - Make up a story problem involving 31 divided by 4 in which the answer is 8.

a. The Standard Division Algorithm
Using the problems below, discuss why each step of the standard division algorithm works. As you discuss each problem, try to identify each step in the algorithm and address questions similar to those in Part Two (The Standards Multiplication algorithm).

72 ÷ 3   252 ÷ 4   8208 ÷ 27

c. Scaffolding Method of Division
The scaffolding algorithm for division is illustrated in the two examples below. This algorithm involves making "guesses" that get you closer and closer to the solution. Therefore, the particular numbers in the examples depend on what guesses are made.

			1
	2		6
	10		20
	60		50
8	576	64	4953
	480		3200
	96		1753
	80		1280
	16		473
	16		384
	0		89
			64
			25

- Make up at least two long division problems and use the scaffolding algorithm to determine the quotient.
- Why does this algorithm work?
- What advantages might this algorithm have in teaching?

18. Solve each of the following using some method other than our standard algorithm.
    a. 34 x 26
    b. 174 ÷ 6

19. Make up a story problem involving 27 ÷ 6 in which the answer is 4. Make up a story problem involving 27 ÷ 6 in which the answer is 5.

Technology

20. In problem #14, you investigated several multiplication algorithms from *Reflections on Number*. Take some time to try to explain why these algorithms work and make sense. After you have done that, go to the following website: http://matti.usu.edu/nlvm/nav/topic_t_1.html
    Once there, scroll down and click on “Rectangle Multiplication: Lattice.”
    This applet allows users to visualize the product of two factors in three different ways: grouping, common, or lattice. Work through each of these representations – and then try to use the visualizations to help you justify or modify the explanations you created above.

21. Go to the web site: http://www.shodor.org/interactivate/activities/egame/index.html and play Estimator 4. Change the tolerance and time. Be sure to only use the addition and multiplication problem types. What role can games such as this one play in the classroom?

Division by Zero

22. Each statement below is true. Explain why each statement is true. (Note: Simply stating a rule does not constitute an explanation of <u>why</u>.)
    c. $0 \div 0$ is undefined.
    d. $6 \div 0$ is undefined.
    e. $0 \div 6 = 0$.

23. Do *Reflections on Number* Section C "Investigating Algorithms" #15-20 (p. 28-29) and Summary Question #25 (p. 32). Write up your responses to #15, 16, 18-20, 25.

Content Combined

24. Addition, subtraction, multiplication and division:
    a. In your own words, what does “addition” mean? What is going on when you add two numbers?
    b. In your own words, what does “subtraction” mean? What is going on when you subtract one number from another?
    c. In your own words, what does “multiplication” mean? What is going on when you multiply two numbers?
    d. In your own words, what does “division” mean? What is going on when you divide one number by another?
    e. Describe in your own words the relationships between addition, subtraction, multiplication, and division. Give an example(s) to illustrate your description.

<u>Readings:</u>

Supplemental readings:

25. Read “Number: What is there to know?” (p. 71 – 114) from *Adding It Up: Helping Children Learn Mathematics*. Available online at www.nap.edu/books/0309069955/html/.

26. Read “Alternative algorithms: Increasing options, reducing errors” by Tamela D. Randolph and Helene J. Sherman from *Teaching Children Mathematics, 7*(8), April 2001.

27. Read the following chapters from *The Teaching and Learning of Algorithms in School Mathematics,* NCTM 1998 Yearbook:
    a. “Alternative Algorithms for Whole-Number Operations,” by William Carroll and Denise Porter (Chapter 14)

b. “The Harmful Effects of Algorithms in Grades 1-4,” by Constance Kanii and Ann Dominick (Chapter 17)
c. “Children’s Invented Algorithms for Multidigit Multiplication Problems,” by Jea-Meen Baek (Chapter 19)

Factors, Multiples, Prime Numbers

<u>Content Covered</u>:
- Composite and Prime numbers
- Factors and multiples
- GCF, LCM

<u>SOLs and NCTM Standards</u>:

Associated SOLs

6.3 The student will

a) find common multiples and factors, including least common multiple and
greatest common factor;

b) identify and describe prime and composite numbers; and
identify and describe the characteristics of even and odd integers.

Associated NCTM Standards

Grades 3 – 5
- describe classes of numbers according to characteristics such as the nature of their factors

Grades 6 – 8
- use factors, multiples, prime factorization, and relatively prime numbers to solve problems

<u>Problems and Activities (by Content)</u>:

Composite and Prime Numbers

1. Answer the following questions below to the best of your ability. You will be revisiting these questions again towards the end of the assignment.
   a. How do you describe a prime number? A composite number? What is the relationship between prime and composite numbers?
   b. Consider F = 151 x 157. Is F a prime number? Explain your decision.
   c. Consider $m(2k + 1)$, where m and k are whole numbers. Is this number prime? Can it ever be prime?

2. Complete the following problems from *Reflections on Number:*
   Section B: Using Prime Factors -- #1-11, 19-20

Factors and Multiples (and more on Prime Numbers)

3. Work through *Skip Counting.* (Explorations, pg. 56-58)

4. Work through *Exploring Factors*. (Explorations, pg. 59-60)

5. Do these problems in "Prime Time":
   a. *Investigation 1:* The Factor Game - Play game (p. 6-9); Problem 1.2 AB and follow-up #1-5
   b. *Inv. 3:* Factor Pairs -- Problem 3.1 and follow-up; Read p. 28; Problem 3.3 ABC
   c. *Inv. 4:* Common Factors and Multiples -- Problem 4.1 ABC and follow-up; Prob 4.3
   d. *Inv 5:* Factorizations -- Read p. 48; Problem 5.2; Problem 5.3 and follow-up #1-4
   e. *Inv 6:* Locker Problem -- Problem 6.1 and follow-up

6. Do these problems in Reflections on Number:
   Section A: Factors and Divisors -- #1-5, 14-21, 25-31

7. Answer these questions after you complete the assigned problems (from #5 and #6) in the *Reflections on Number* and *Prime Time* books.
   a. Compare the general approach of these two textbooks. Address some of these questions in your response:
      i. What is mathematically similar/different in the two books?
      ii. What is similar/different about how a teacher might use these books in the classroom?
      iii. What is similar/different about how students work on the problems and activities in the two books?
      iv. If you were planning to use one of these books in your classroom, which one would you choose? Why?

   b. Consider the Changing Positions Problem (in *Reflections on Number*) and the Locker Problem (in *Prime Time*).
      1. How are these two problems related? What features of the two problems are similar? What features are different?
      2. Did you find one of the problems easier to solve than the other? Why or why not?
      3. Why do you think the Changing Positions Problem appears early in the *Reflections on Number* book, but the Locker Problem appears at the very end of the *Prime Time* book? (Why did the textbook writers order things this way?)
      4. Which placement makes more sense to you? Why? (If you were writing a textbook, or teaching with one of these books, would you prefer to begin with this type of problem, or end with it, or something else?)

Greatest Common Factors (GCF) and Least Common Multiples (LCM)

8. Kathy and Pam like to walk around the track for exercise. Since they walk at different rates, they start off together but do not stay together during the walk. They always begin their walk from a bench that marks their starting point. Kathy takes 6 minutes to go around the track one time and Pam takes 9 minutes.
   (a) How many minutes would Kathy and Pam have to be walking before they find themselves at the bench at the same time?
   (b) Explain your answer using *either* the word <u>multiple</u> or the word <u>factor</u>.

9. Marcel wants to plant some small flower beds in his yard. He wants each flower bed to contain the same number of marigolds, and he wants each bed to contain the same number of daisies. He has 120 marigold seeds and 90 daisy seeds.
   (a) What is the largest number of flower beds Marcel can make with these seeds?
   (b) Explain your answer using *either* the word <u>multiple</u> or the word <u>factor</u>.

Content Combined:

10. Answer the following questions:
    a. Find a number greater than 500 that has an odd number of factors.
    b. Find the GCF (greatest common factor) of 100 and 124.
    c. Find the LCM (least common multiple) of 20 and 24.
    d. Which numbers between 130 and 140 are prime numbers?
    e. What mathematical understandings would students need to have to be able to answer the questions above (a-d)? Are these "good" questions? How could you expand on these questions to generate classroom discussion about the mathematics inherent in the questions?

11. At the Bubble Gum Factory, lengths of gum are stretched to larger lengths by putting them through stretching machines. There are 100 stretching machines, numbered 1 through 100. Machine 1 does nothing to a piece of gum; machine 2 stretches pieces of gum to twice their original length; machine 3 triples the length and so forth. So, machine 23, for example, will stretch a piece of gum to 23 times its original length.
    An order has just come in for a piece of bubble gum 26 inches in length. The factory has pieces of gum that are only 1 inch in length, and machine number 26 is broken. Is there any way to create a piece of bubble gum 26 inches in length by using other machines? Some of the machines in the factory are unnecessary because combinations of other machines could be used instead. Figure out which machines are actually unnecessary.

    a. What machines would be necessary to get the following lengths: 15? 28? 36? 65? 84?

b. For each of the lengths in question 1, what other machines could have been used that were unnecessary?
c. Which lengths between 1 and 100 would come out if the bubble gum went through five machines and all five machines were necessary ones?
d. Which lengths between 1 and 100 require the greatest number of necessary machines? How did you figure out your answer?

Readings:

Supplemental readings:

12. Read "Understanding Primes: The Role of Representation," from the *Journal for Research in Mathematics Education*. (Note: Do not worry about reading the participants and setting section or the data collection and analysis sections too closely).
    a. What do the authors mean when they talk about transparent and opaque representations in mathematics? Give an example from your own teaching.
    b. Did reading this article focused on preservice teachers understandings of primes help you develop any new ideas about this concept? If so, how did that happen and what new ideas did you come up with?
    c. The authors of this article talk briefly about the questions they used to probe understanding of the preservice teachers in this study. Could you use some of these questioning methods in your own classroom? If so, how?

Fraction Sense

<u>Content Covered</u>:

- Fraction Sense
- Part-Whole Meaning of Fractions
- Mixed Numbers

<u>SOLs and NCTM Standards</u>:

Associated SOLs

3.5 The student will
  a) divide regions and sets to represent a fraction; and
  b) name and write the fractions represented by a given model (area/region, length/measurement, and set). Fractions (including mixed numbers) will include halves, thirds, fourths, eighths, and tenths.

3.6 The student will compare the numerical value of two fractions having like and unlike denominators, using concrete or pictorial models involving areas/regions, lengths/measurements, and sets.

3.11 The student will add and subtract with proper fractions having like denominators of 10 or less, using concrete materials and pictorial models representing areas/regions, lengths/measurements, and sets.

4.2 The student will
  a) identify, model, and compare rational numbers (fractions and mixed numbers), using concrete objects and pictures;
  b) represent equivalent fractions; and relate fractions to decimals, using concrete objects.

5.2 The student will
  a) recognize and name commonly used fractions (halves, fourths, fifths, eighths, and tenths) in their equivalent decimal form and vice versa; and
  b) order a given set of fractions and decimals from least to greatest. Fractions will include like and unlike denominators limited to 12 or less, and mixed numbers.

6.4 The student will compare and order whole numbers, fractions, and decimals, using concrete materials, drawings or pictures, and mathematical symbols.

6.5 The student will identify, represent, order, and compare integers**.**

6.7 The student will use estimation strategies to solve multistep practical problems involving whole numbers, decimals, and fractions (rational numbers).

Associated NCTM Standards

Grades 3 – 5

- develop understanding of fractions as parts of unit wholes, as parts of a collection, as locations on number lines, and as divisions of whole numbers

- use models, benchmarks, and equivalent forms to judge the size of fractions
- recognize and generate equivalent forms of commonly used fractions, decimals, and percents
- develop and use strategies to estimate computations involving fractions and decimals in situations relevant to students' experiences

Grades 6 – 8

- compare and order fractions, decimals, and percents efficiently and find their approximate locations on a number line

Problems and Activities (by Content):

Fraction Sense

1) Work through "Which is More?" from *Thinking Rationally About Fractions, Decimals and Percents* (www.pen.k12.va.us/VDOE/Instruction/Math/FractionsDecimalsPercent.pdf).
   Complete the activities on page 122 (instead of reading through the lesson plan first). After you work through the activities, read the lesson plan and discuss how you might adapt this lesson to use in their own classrooms.

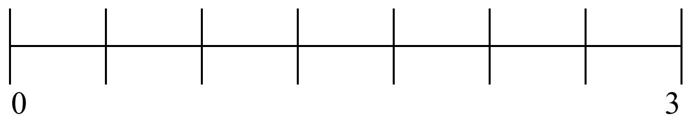
2) Work through *Developing Fraction Sense.* (Explorations, pg. 61)

3) Work in your groups to determine a variety of ways to solve the following problems. What ideas about patterns and fractions would students learn when doing these problems?
   a. The cookie monster sneaks into the kitchen and eats half a cookie; on the second day he comes in and eats half of what remains of the cookie from the first day; on the third day he comes in and eats half of what remains from the second day. If the cookie monster continues this process for seven days, how much of the cookie has he eaten? How much is left? If the process continues, will he ever eat all the cookies?
   b. A cricket is on the number line at the point labeled 0. She wants to get to the point labeled 1, but she can hop only half the remaining distance each time. Where on the number line is the cricket after one hop? Two hops? Seven hops? Does the cricket ever get to the point labeled 1?

4) What happens to the fraction $1/n$ as n gets larger and larger or as n gets smaller and smaller? Talk in your groups about ways to investigate this question in the classroom.

5) Find a fraction, if possible, between 1/6 and 1/7.

6) A number line from 0 to 3 is broken into seven equal segments. Name the fractions at each mark on the line. Choose one of the marks and explain how you determined its value.



Technology

7) Go to the website listed here and play "fraction sorter": http://www.shodor.org/interactivate/activities/fracsorter/index.html
Change both the number of fractions to order and the shape as you play with this applet.
What role can this applet play in developing fraction sense?

8) Look at the fraction applet at the following web address: http://arcytech.org/java/fractions/. How could you use this applet with your students? What understandings could students develop about fractions while working with this applet?

Part-Whole Meaning of Fractions

9) Work through *Exploring the Part-Whole Meaning of Fractions* (Explorations, pg. 62)

10) If this group of triangles is 11/5 of one whole, how many triangles are in one whole?

Δ Δ Δ Δ Δ Δ Δ Δ Δ Δ Δ

11) Five identical pizzas are to be shared fairly among six friends. What fraction of one pizza will each friend get?

Technology

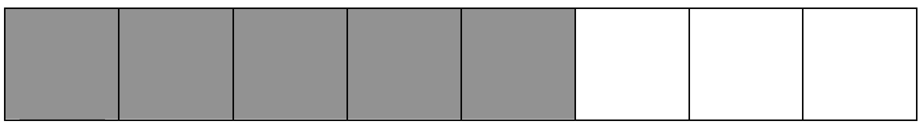
12) Work through problems 1-3 from *Exploring Fractions with Pattern Blocks* (Explorations, pg. 63-65). After working through this activity, adapt the activity for your particular students. Would you keep it the same, add more problems, shorten the assignment, change the level of difficulty, create an introduction…?

Mixed Numbers

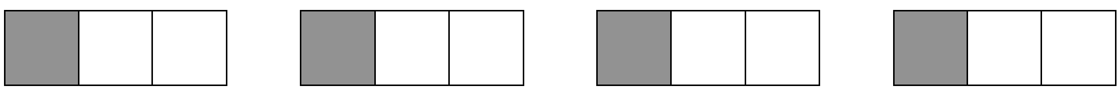
13) There is a rule for converting a "mixed number" like 34/9 into a fraction by multiplying the 9 and the 3, adding the 4, and writing the result over 9: 31/9.
    a. Explain why this rule works. Include a picture with your explanation.
    b. Explain a method for converting 31/9 to 3 4/9 and explain why it works.

14) For each of the examples below, indicate what you think the student was thinking and how you might help the student to more accurately represent the fraction shown.

(a) Melissa's drawing of 5/4:



(b) Juan's drawing of 4/3:



<u>Readings</u>:

Case studies:

15) Read "Problem of the Week: Bounce, Bounce" from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?* (p. 62-68). Look back at Derrick's and Leslie's work. What trouble did they both have? How would you help these two students?

16) Read "Six Hours Isn't One-Sixth of a Day" from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?* (p. 110-111).

17) Read "Six-Tenths or Four-Fifths of a Dollar?" from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?* (p. 45-47).

18) Read "Thirteen Can't Fit Over Twelve" from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?* (p. 55-59).

Supplemental readings:

19) Briefly read through "The Strands of Mathematical Proficiency," chapter 4 from *Adding it Up,* to familiarize yourself with what the authors have defined as the five strands of mathematical proficiency. Then read, and focus more closely on, "Developing Proficiency with Other Numbers," chapter 7 from *Adding it Up.* Both chapters are available online at www.nap.edu/books/0309069955/html/.
    a. Throughout chapter 7, the authors make frequent statements about how students understand and make sense of specific topics in mathematics. In general, do you agree with these authors' assertions? Why or why not?
    b. Select one or two statements in particular that the authors made about typical student understanding and compare that to what you have experienced with your own students understanding of rational numbers.
    c. Discuss one idea (a teaching method, a common misconception, a mathematical concept) that you feel you learned through reading this article.

20) The article, "Developing Children's Understanding of the Rational Numbers: A New Model and an Experimental Curriculum," focuses on a new rational numbers curricula developed by the authors. In order to determine the new curricula's effect on children's understanding, the authors interviewed two groups of students (one group using the new curricula, one group continuing to use a more traditional curricula) before and after instruction to measure changes in understanding. Before reading this article, answer the following sample of questions from the interviews yourself – it will provide a nice context as you read students answers to these same questions. To the extent possible, try to reason through how you would come to an answer for each question instead of simply calculating an answer using an algorithm and writing a numerical answer down next to each problem. Write down as much of your ideas as you can – it will be interesting to go back and evaluate your own reasoning processes after reading the article.

    a. A student told me that 7 is 3/4 of 10. Is it? (again, instead of just answering these questions, reason through the problem and provide justification for your answer)
    b. What is 65% of 160?
    c. Can you think of a number that lies between decimal 3 and decimal 4?
    d. Draw a picture to show which is greater, 2/3 or 3/4.
    e. Could these be the same amount: 0.06 of 1/10 and 0.6 of 1/100?
    f. A CD is on sale. It has been marked down from $8.00 to $7.20. What is the discount as a percentage of the original price?
    g. What is 1/8 as a decimal?

21) Read "Developing Children's Understanding of the Rational Numbers: A New Model and an Experimental Curriculum," from the *Journal for Research in Mathematics Education (1999, Vol. 20, No. 2).*

a. The authors of this article discuss a new curricula they have developed that they hope will help children to develop better overall conceptions of rational numbers. Describe the authors' rationale for why they believe this curriculum will work.
b. What do you think of this approach to teaching rational numbers?
c. How does this approach, specifically their focus on the sequence of topics, compare to your own approach to teaching rational numbers?
d. Relying on your own experiences teaching this topic in the past, how do you think this approach would work in your own classroom?

Adding & Subtracting Fractions

Content Covered:

- Adding and Subtracting Fractions
- Area Model for Fractions

SOLs and NCTM Standards:

Associated SOLs

3.5 The student will

a) divide regions and sets to represent a fraction; and

b) name and write the fractions represented by a given model (area/region, length/measurement, and set). Fractions (including mixed numbers) will include halves, thirds, fourths, eighths, and tenths.

3.6 The student will compare the numerical value of two fractions having like and unlike denominators, using concrete or pictorial models involving areas/regions, lengths/measurements, and sets.

3.11 The student will add and subtract with proper fractions having like denominators of 10 or less, using concrete materials and pictorial models representing areas/regions, lengths/measurements, and sets.

4.2 The student will

a) identify, model, and compare rational numbers (fractions and mixed numbers), using concrete objects and pictures;

b) represent equivalent fractions; and relate fractions to decimals, using concrete objects.

4.9 The student will

a) add and subtract with fractions having like and unlike denominators of 12 or less, using concrete materials, pictorial representations, and paper and pencil;

b) add and subtract with decimals through thousandths, using concrete materials, pictorial representations, and paper and pencil; and

c) solve problems involving addition and subtraction with fractions having like and unlike denominators of 12 or less and with decimals expressed through thousandths, using various computational methods, including calculators, paper and pencil, mental computation, and estimation.

5.7 The student will add and subtract with fractions and mixed numbers, with and without regrouping, and express answers in simplest form. Problems will include like and unlike denominators limited to 12 or less

6.6 The student will

a) solve problems that involve addition, subtraction, multiplication, and/or division with fractions and mixed numbers, with and without regrouping, that include like and unlike denominators of 12 or less, and express their answers in simplest form; and

b) find the quotient, given a dividend expressed as a decimal through thousandths and a divisor expressed as a decimal to thousandths with exactly one non-zero digit.

Associated NCTM Standards

Grades 3 – 5

- use visual models, benchmarks, and equivalent forms to add and subtract commonly used fractions and decimals

Grades 6 – 8

- understand the meaning and effects of arithmetic operations with fractions, decimals, and integers
- develop and analyze algorithms for computing with fractions, decimals, and integers and develop fluency in their use

<u>Problems and Activities (by Content)</u>:

Adding and Subtracting Fractions

1) Work through the all of the activities in Investigation 4 from *Bits and Pieces II*. (See Reading list below for article related to these problems)

2) Explain why it makes sense to add fractions according to the rule:

$$\frac{A}{B}+\frac{C}{D}=\frac{A*D+C*B}{B*D}$$

Technology

3) Work through problem 4 from *Exploring Fractions with Pattern Blocks* (Explorations, pg. 66).

Area Model for Fractions

4) Work through the "Playground Problem" (p. 80) from *Thinking Rationally About Fractions, Decimals and Percents* (VA DOE 2002: www.pen.k12.va.us/VDOE/Instruction/ Math/FractionsDecimalsPercent.pdf).

5) Work through *Area Models for Fractions* (Explorations, pg. 67-68). Included with the handout for this activity is geoboard dot paper. If you have access to geoboards, use those instead.

6) Work through "All Cracked Up – Area and Fractions with Tangrams" (p. 10) from *Thinking Rationally About Fractions, Decimals and Percents* (VA DOE, 2002). If tangram manipulatives are available, you might want to use those instead.

<u>Readings</u>:

Case studies:

7) Read “Two Green Triangles” from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?* (p. 86-88).

8) Read “Beans, Rulers, and Algorithms” from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?* (p. 7-9).

9) Read “The Beauty of Math” from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?* (p. 119-121).

Supplemental readings:

10) Read “Developing Algorithms for Adding and Subtracting Fractions” from *The Teaching and Learning of Algorithms in School Mathematics*. This chapter focuses on three problems that come directly from *Bits and Pieces II* – as you discuss each problem above as a whole class, read the relevant parts of the chapter and discuss the sample student work that is included.

11) Read “Letting Fraction Algorithms Emerge through Problem Solving” from *The Teaching and Learning of Algorithms in School Mathematics*. Write down one new idea you gained from reading the chapter.

12) Read “Unpacking the Nature of Discourse in Mathematics Classrooms” by Knuth and Peressini and “One Fraction Problem, Many Solution Paths” by Yang and Reys from *Mathematics Teaching in the Middle School*. Answer the following questions.
    a. Describe in your own words the difference between what Knuth and Peressini refer to as univocal and dialogic discourse. Which type of discourse do you feel develops most often in your own classroom?
    b. Yang and Reys present a fraction problem in their article that they feel “stimulates thinking and encourages multiple approaches” and then illustrate the important role of the teacher in choosing questions. Take some time now to create a problem that you feel stimulates thinking and encourages multiple approaches (given our current focus, create a problem dealing with rational numbers, in any way). After you create a problem, create a list (similar to the one at the bottom of the second page of the Yang and Reys article) that outlines student capabilities that might be revealed through the use of this problem.
    c. Using both articles as a guide, create a short dialogue between you and your students that might arise as students work through the problem. Keep in mind the ideas of dialogic and univocal discourse presented in the first article.

Multiplying Fractions

<u>Content Covered</u>:
Multiplying fractions

<u>SOLs and NCTM Standards</u>:

Associated SOLs

6.6 The student will solve problems that involve addition, subtraction, multiplication, and/or division with fractions and mixed numbers, with and without regrouping, that include like and unlike denominators of 12 or less, and express their answers in simplest form; and

Associated NCTM Standards

Grades 6 – 8

- understand the meaning and effects of arithmetic operations with fractions, decimals, and integers
- develop and analyze algorithms for computing with fractions, decimals, and integers and develop fluency in their use

<u>Problems and Activities (by Content)</u>:

Multiplying Fractions

1) Explain why $\frac{4}{5} = \frac{4*3}{5*3} = \frac{12}{15}$. In what context does doing something like this make sense?

2) Explain why $\frac{1}{10} \times 3 = 3 \div 10$.

3) Illustrate (in as many ways as you can think of) what is going on in the following multiplication problems by drawing pictures.

(a) 2/3 x 1/2	(g) 1 x 1/2
(b) 1/2 x 1/3	(h) 1/3 x 1/6
(c) 5/6 x 1/2	(i) 4/3 x 3/2
(d) 1/2 x 2/3	(j) 1/2 x 1/3
(e) 3/4 x 6	(k) 5/6 x 1/3
(f) 5/8 x 1/3	(l) 4/3 x 6

4) Work through the problems and activities from Investigation 5 in *Bits and Pieces II.*

5) Work through the following problems. Come up with answers without using any traditional algorithms and try to represent each story problem with a picture.

a. At the supermarket, potatoes were bagged in 3/4-pound bags. Mom bought 3 bags of potatoes. How many pounds of potatoes did Mom buy?
b. Mrs. Smith has 120 books in her fourth-grade classroom, 4/5 of the books are fiction. How many books are fiction?
c. Before the new semester, all the notebooks at the local store are discounted by 1/4. A notebook originally costs $0.96. How much do you save on one notebook if you buy it today?
d. Julie bought 4/5 of a yard of material for her class project. Later, she found that she needed only 3/4 of the materials. How much material did Julie use for her project?
e. Cabbage costs $0.39 a pound. Julie bought 3 1/3 pounds of cabbage to prepare her dish. How much did she pay for the cabbage?

6) Fraction word problems:
    a. Make up several interesting story problems in which the following could reasonably arise: 3/4 x 1/8
    b. Discuss in your groups what is important to understand about the process of fraction multiplication in order to create these word problems. In what ways would this type of activity be beneficial to your own students? What would this help them to understand about fractions?

7) A student in your class, Miranda, says that

$$3\frac{2}{3}\times 2\frac{1}{5} = 3\times 2 + \frac{2}{3}\times\frac{1}{5}$$

Is Miranda correct? If so, explain her reasoning. If not, explain how you might work with what Miranda has already written and help her modify it to get the correct answer.

8) Explain why $\frac{2}{7} * \frac{3}{4} = \frac{2*3}{7*4}$.

<u>Readings</u>:

Case studies:

9) Read "Take One-Third" from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?* (p. 4-6).

10) Read "There's No One-Half Here" from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?* (p. 10).

11) Read “I Still Don’t See Why My Way Doesn’t Work” from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?* (p. 28-29).

Supplemental readings:

12) Read “Multiplying Fractions” from *Teaching Children Mathematics*. This article highlights student solutions to the problems you just attempted –discuss your solutions in relation to those presented in the article.

Dividing Fractions

Content Covered:
- Dividing Fractions
- Flip and Multiply

SOLs and NCTM Standards:

Associated SOLs

6.6 The student will

a) solve problems that involve addition, subtraction, multiplication, and/or division with fractions and mixed numbers, with and without regrouping, that include like and unlike denominators of 12 or less, and express their answers in simplest form; and

b) find the quotient, given a dividend expressed as a decimal through thousandths and a divisor expressed as a decimal to thousandths with exactly one non-zero digit.

Associated NCTM Standards

Grades 6 – 8
- understand the meaning and effects of arithmetic operations with fractions, decimals, and integers
- develop and analyze algorithms for computing with fractions, decimals, and integers and develop fluency in their use

Problems and Activities (by Content):

Dividing Fractions

1) We can all agree that $2 \div 5 = \frac{2}{5}$. But do these two representations have different meanings? Try to create a word problem or particular context in which you highlight the fact that both expressions represent the same amount, but are described and used in different ways.

2) Fraction word problems:
Make up an interesting story problem in which the following could reasonably arise: 2/3 divided by 1/4 (if you are unsure right now, that's okay, we will be working on this type of problem during our next class meeting – if you are having trouble, try to write out where your confusion lies)

3) Which of the following are solved by the division problem $\frac{3}{4} \div \frac{1}{2}$? For those that are, try to determine the interpretation of division is used (e.g. how many groups, how many in each group…). For those that are not, determine how to solve the problem, if it can be solved.

   a. $\frac{3}{4}$ of a bag of gummi worms make $\frac{1}{2}$ a cup. How many cups of gummi worms are in one bag?
   b. $\frac{3}{4}$ of a bag of gummi worms make $\frac{1}{2}$ a cup. How many bags of gummi worms does it take to make one cup?
   c. You have $\frac{3}{4}$ of a bag of gummi worms and a recipe that calls for $\frac{1}{2}$ of a cup of gummi worms. How many batches of your recipe can you make?
   d. You have $\frac{3}{4}$ of a cup of gummi worms and a recipe that calls for $\frac{1}{2}$ of a cup of gummi worms. How many batches of your recipe can you make?
   e. If $\frac{3}{4}$ of a pound of candy costs $\frac{1}{2}$ of a dollar, then how many pounds of candy should you be able to buy for 1 dollar?
   f. If you have $\frac{3}{4}$ of a pound of candy and you divide the candy in $\frac{1}{2}$, then how much candy will you have in each portion?
   g. If $\frac{1}{2}$ of a pound of candy cost \$1, then how many dollars should you expect to pay for $\frac{3}{4}$ of a pound of candy?

4) Suppose you are talking to someone who knows about fractions but does not know how to multiply or divide them. Explain to this person what $\frac{4}{3} \div \frac{1}{2}$ means, and show the person a way to determine the answer to this division problem with diagrams and logical thinking as opposed to a procedure.

Technology

5) Work through *Dividing Fractions* (Explorations, pg. 69-70). After working through the activities and problems, adapt the activity for your particular students. Would you keep it the same, add more problems, shorten the assignment, change the level of difficulty, create a context relevant to your community…?

Flip and Multiply

6) Solve the problem below by using the "flip and multiply" method for fraction division.
   Melvin has 11 *yards of cloth to make some new Little League baseball uniforms. Each uniform requires* $1\frac{1}{2}$ *yards of cloth. How many uniforms can he make?*
   a. What is the remainder in this problem? What is the meaning of the remainder?
   b. Now solve the problem by drawing a picture and using the picture to solve the problem.
   c. What is the remainder this time? What is the meaning of the remainder?
      a. Why are there two different remainders?
      b. Make up two different questions (for this problem situation) that have the two different remainders as their answers.

7) Summarize, in your words, why the "flip and multiply" rule for fraction division makes sense. Your explanation should include pictures and at least one example.

Fraction Sense Revisited

8) Suppose that $a > 1$, $0 < b < 1$, and $0 < c < 2$. Complete each sentence by filling in the blank with <, =, >, or CT (can't tell). How would this activity encourage students to make generalizations about operations involving fractions of various sizes?

   a. $a \bullet b$ ☐ a
   b. $b \bullet c$ ☐ b
   c. $a \bullet b \bullet c$ ☐ a
   d. $a \div b$ ☐ a
   e. $a \div c$ ☐ a
   f. $b \div c$ ☐ b
   g. c/c ☐ c
   h. b^2 ☐ b

Readings:

Supplemental readings:

9) Read "A Constructed Algorithm for the Division of Fractions" from *The Teaching and Learning of Algorithms in School Mathematics.* What do you think of the ideas about fraction division presented in each chapter?

10) "Dividing Fractions by Using the Ratio Table" from *The Teaching and Learning of Algorithms in School Mathematics.* What do you think of the ideas about fraction division presented in each chapter? Be prepared to discuss the ideas raised in these chapters in class.

Place Value

Content Covered:

- Base 10 and Other Number Systems
- Working in Different Bases
- Modular Art

SOLs and NCTM Standards:

Associated SOLs

3.1 The student will read and write six-digit numerals and identify the place value for each digit.

Associated NCTM Standards

Grades 3 – 5

- understand the place-value structure of the base-ten number system and be able to represent and compare whole numbers and decimals

Problems and Activities (by Content):

Base 10 and Other Number Systems

1) Work through *Alphabetia* (Explorations, pg. 71).

2) Compare your system (from the *Alphabetia* exploration in #1) to our system. What are the similarities? Differences?

3) Explain how to change a number from the base system you developed in 1) above to the base 10 system.

4) Explain how to change a number form the base 10 system to the base system you developed in 1) above.

Working in Different Bases

5) We use different bases in many situations. Use number bases to answer the following questions:
   i) Change 52 days to weeks and days.
   ii) Change 54 months to years and months.
   iii) Change 158 hours to days and hours
   iv) Change 55 inches to feet and inches.

   Can you come up with other situations where different numbers bases arise?

6) Suppose you need to purchase 1,000 name tags and can buy them by the gross (144), the dozen (12) or individually. The name tags cost $.50 each, $4.80 per dozen and $50.40 per gross. How should you order to minimize the cost?

7) Convert the following:
   i) Change \$4.59 into quarters, nickels and pennies. (Which other problem does this relate to? Why were these specific coins chosen?)
   ii) Suppose you have two quarters, four nickels, and two pennies. Use base five to write a numeral to indicate your financial status.
   iii) Change \$8.34 into the smallest number of coins consisting of quarters, nickels and pennies.

8) Assume that today is Monday (day 2). Determine the day of the week it will be at the end of each of the following periods. (Assume no leap years.)
   a. 24 days   b. 155 days   c. 365 days   d. 2 years

9) Suppose you make six round trips to visit a sick aunt and wish to record your mileage to the nearest mile. You forget the original odometer reading, but you do remember that the units digit has increased by 8 miles. What are the possible distances between your house and your aunt's house?

10) Use the set {0, 1, 2, 3, 4, 5, 6} for the following:
   i) Make a table for addition and multiplication using base 7. (mod 7)
   ii) Is the set closed for addition?
   iii) Is the set closed for multiplication?
   iv) Does the commutative property hold for either?

11) Which of the following statements are true? Explain.

   A. i) 5 + 8 = 1 (mod 6)   ii) 4 + 5 = 1 (mod 7)
   B. i) 5 = 53 (mod 8)   ii) 102 = 1 (mod 2)
   C. i) 47 = 2 (mod 5)   ii) 108 = 12, (mod 8)

12) Complete *Alphabetia Revisited* (Explorations, pg. 72).

13) Check Digits. Bar codes and associated numbers make up the Universal Product Code (UPC). Each bar code is usually made up of a 12-digit number. The first six digits encode information about the manufacturer and the next five encode information about the product. The last digit is called the check digit. The check digit provides a way to determine if a UPC number is incorrect. Here's how it works. If the first 11 digits of the UPC number are lined up, we multiply every other number by 3 and then add up all the numbers. The number, 071662040123, is a UPC number from a box of colored pencils. Taking the first 11 digits, we have:

$$(3 \times 0) + 7 + (3 \times 1) + 6 + (3 \times 6) + 2 + (3 \times 0) + 4 + (3 \times 0) + 1 + (3 \times 2) = 47$$

We now select the check digit to be the number from 0 to nine so that when we add the above sum to it, we get an answer equivalent to 0 mod 10. For our

example above, we want a number, call it c, so that 47 + c = 0 (mod 10) or equivalently,
7 + c = 0 (mod 10). So in our case, c = 3. Use this information to determine which of the following are valid UPC codes.

A. Spaghettios:
   0 51000 02562 4
   0 51000 02526 4
   0 51000 02526 5
B. Progresso
   0 41196 91912 1
   0 52010 00121 2
   0 05055 00505 3

Try this on different UPCs of products at home. Can you see what kinds of errors the check digit will not catch?

14) Work through *Lisa Simpson Age Guesser* (Explorations, pg. 73-74) or *The Fruity Magic Trick* (Explorations, pg. 75).

Technology

15) Clock Arithmetic. Suppose your watch says 6:00 and you are to meet your friend at the mall in 28 hours. What time will your watch say when you get to the mall? We live in a 24 hour world, go to the website: http://www.shodor.org/interactivate/activities/clock1/index.html and use the clock to explore different hour worlds:
    i. What is the clock time for 22 hours on a 12 hour world? On a 10 hour world?
    ii. What is the clock time for 54 hours on a 6 hour world? On a 9 hour world? On a 17 hour world?
    iii. It is 8 o'clock on our 22 hour world. What time is it on a 3 hour world? On a five hour world?
    iv. It is 15 o'clock on our 31 hour world? What time is it on a 3 hour world? On a five hour world?
    v. Now, can you find two different worlds that 78 hours is 6 o'clock? 4 o'clock? Explain.
    vi. Now, can you find two different worlds that 16 hours is 4 o'clock? Explain.
    vii. Our school is on a twelve hour day but the military is on a 24 hour clock. What do you think that schools use one clock and the military uses a different? What are the advantages and disadvantages of each?

16) Relate the Clock Arithmetic problems to your new way of counting in #1.

17) Go to the website: http://illuminations.nctm.org/lessonplans/6-8/bdays/index.html and work through Birthdays and the Binary System. Be sure to go through the construction of the cards. Compare this problem to the Lisa Simpson and Fruity Magic Trick explorations. What kinds of mathematics is involved in problems such as these?

18) On the web, go to: http://arcytech.org/java/b10blocks/b5blocks.html

The applet allows you to work in base 5. Complete different addition and subtraction problems in base 5 using this applet.

19) Using the web site in activity # 18, think of a way to represent multiplication and division in different bases. If you have trouble, the website includes instructions for each. Be prepared to demonstrate your methods.

Modular Art

20) Complete a multiplication table for modulo 12 (using 12 for zero). On the circle (with twelve equally spaced dots on following pages) number the dots from 1 to 12. Pick a row in your table and connect the dots with a line segment. For example, the entries of the third row are:

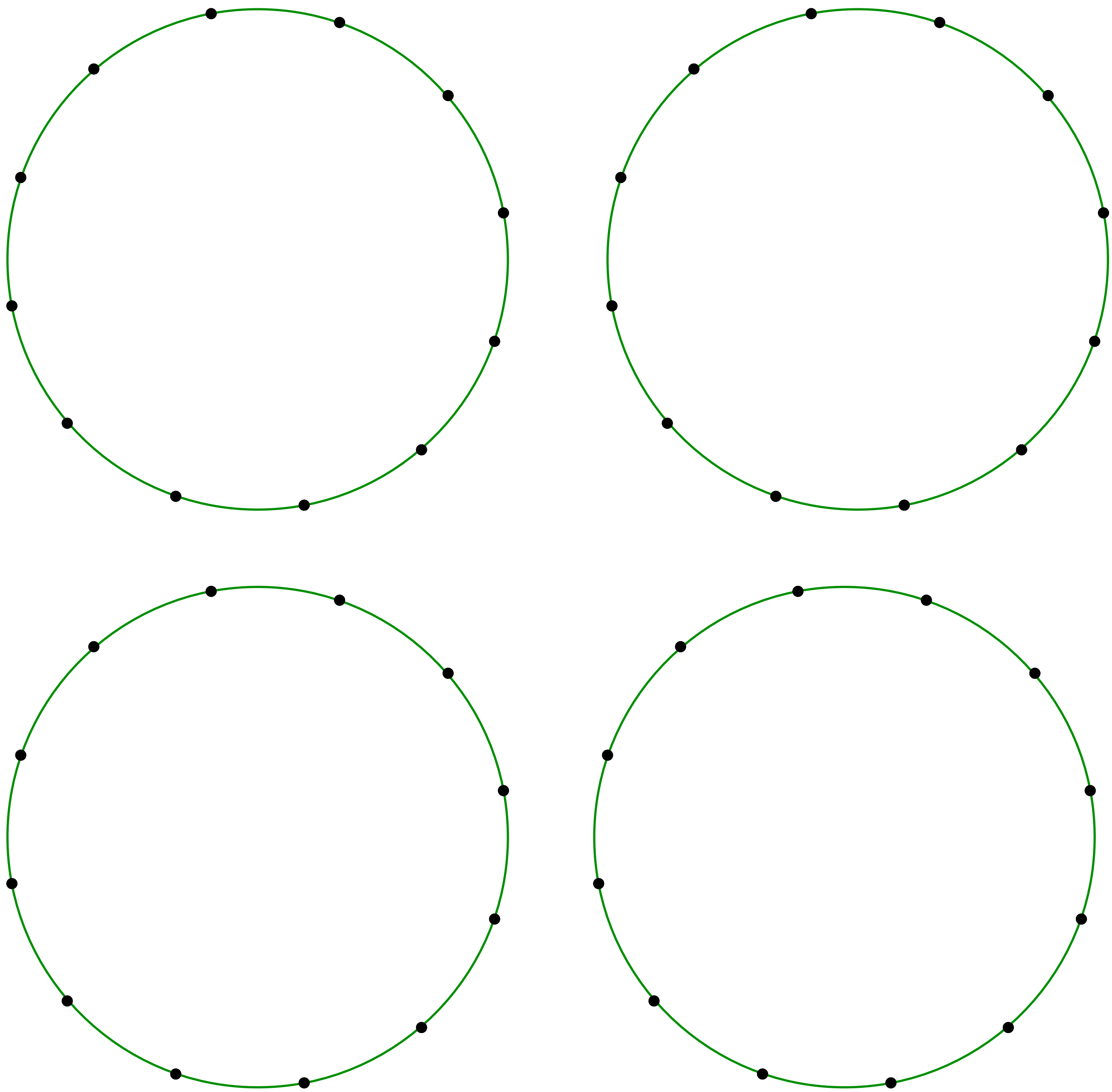
X	1	2	3	4	5	6	7	8	9	10	11	12
3	3	6	9	12	3	6	9	12	3	6	9	12

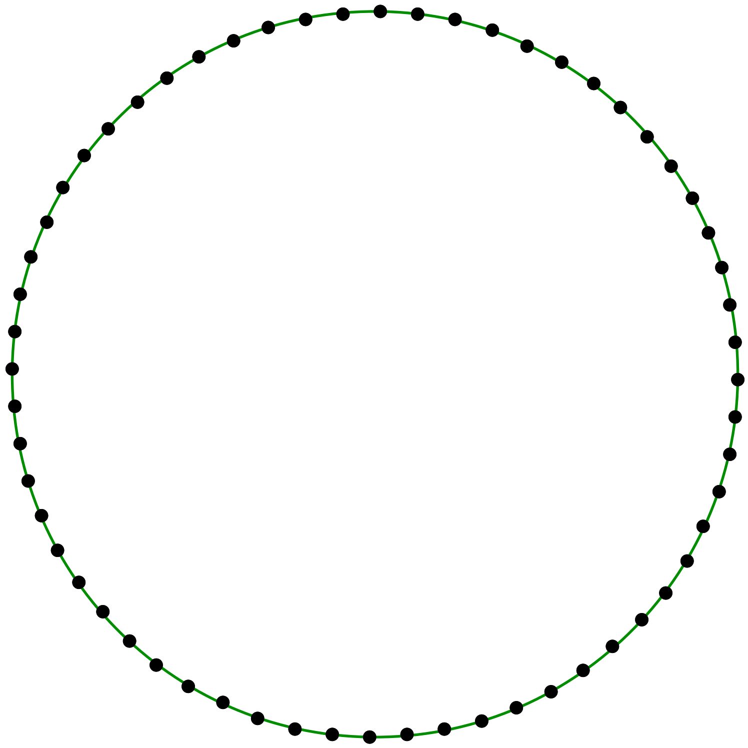
On your circle, there would be a segment connecting the points 1 and 3, 2 and 6 and so on. Once you have the hang of things, try other rows.
Mod 30 will give some interesting results, try row 2 for starters! (circle on following pages)

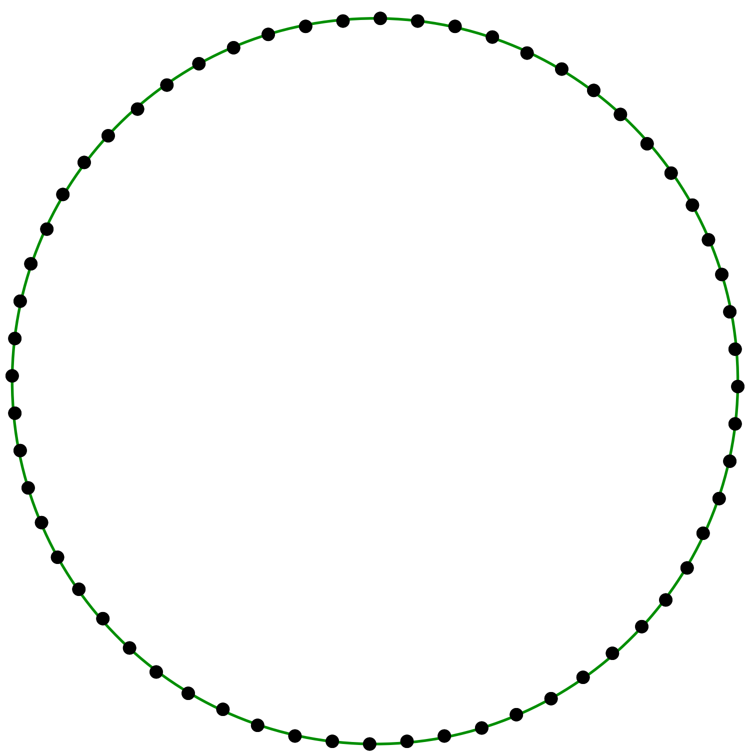
Readings:

Supplemental readings:

21) Modular Art: Many interesting designs can be created using patterns based on modular arithmetic. Read "Using Mathematical Structures to Generate Artistic Designs" by Sonia Forseth and Andrea Price Troutman, *The Mathematics Teacher*, May 1974, pp. 393-398.







Decimals

<u>Content Covered</u>:

- Decimal Sense
- Repeating/Terminating Decimals
- Operating with Decimals

<u>SOLs and NCTM Standards</u>:

Associated SOLs

3.7 The student will read and write decimals expressed as tenths and hundredths, using concrete materials and models.

3.12 The student will add and subtract with decimals expressed as tenths, using concrete materials, pictorial representations, and paper and pencil.

4.4 The student will
  a) read, write, represent, and identify decimals expressed through thousandths;
  b) round to the nearest whole number, tenth, and hundredth; and compare the value of two decimals, using symbols (<, >, or =), concrete materials, drawings, and calculators.

5.1 The student will
  a) read, write, and identify the place values of decimals through thousandths;
  b) round decimal numbers to the nearest tenth or hundredth; and
  c) compare the values of two decimals through thousandths, using the symbols >, <, or =.

5.2 The student will
  a) recognize and name commonly used fractions (halves, fourths, fifths, eighths, and tenths) in their equivalent decimal form and vice versa; and
  b) order a given set of fractions and decimals from least to greatest. Fractions will include like and unlike denominators limited to 12 or less, and mixed numbers.

5.5 The student will find the sum, difference, and product of two numbers expressed as decimals through thousandths, using an appropriate method of calculation, including paper and pencil, estimation, mental computation, and calculators.

5.6 The student, given a dividend expressed as a decimal through thousandths and a single-digit divisor, will find the quotient.

Associated NCTM Standards

Grades 3 – 5

- recognize and generate equivalent forms of commonly used fractions, decimals, and percents
- develop and use strategies to estimate computations involving fractions and decimals in situations relevant to students' experiences

Grades 6 – 8
- compare and order fractions, decimals, and percents efficiently and find their approximate locations on a number line

<u>Problems and Activities (by Content)</u>:

Decimal Sense

1. In the book titled *Math Matters: Understanding the Math you Teach,* the authors, Suzanne Chapin and Art Johnson state the following:
   > "Few teachers have difficulty performing computations with decimals. However, when asked to state important concepts related to decimals, many of us find it hard to articulate even the most fundamental ideas. As a result, instruction in many classrooms focuses on developing students' computational skills rather than on helping students develop a conceptual understanding of quantity, order, and equivalence related to decimal quantities. Looking at some of the "big ideas" related to decimals will help us come to a more focused understanding of some of the mathematical relationships decimals represent." (p. 98)

   What do you think are the big ideas related to decimals? Revisit this question again after working through the activities listed below.

2. Discuss and decide in groups whether the zeros in each of the following numbers are important.

37.0	209.00	300	6.003	500.09
508	1.040	2.0	7.2080	10
9.600	8.000	4.50	10.0	4.0700

3. In Bits and Pieces II, work through problem 6.1.

4. Mary Lou says that 50 times 4.68 is the same as .50 times 468, so she can just take half of 468, which is 234. Can she do this? How could she find 500 times 8.52 in a similar way? Explain.

Technology

5. Work through *Decimals* (Explorations, pg. 76-78)

Repeating/Terminating Decimals

6. Your objective in this activity is to find out whether there is a way to tell whether a fraction can be written using a terminating or repeating decimal. Convert each of the fractions below to a decimal by dividing the numerator by the denominator. This will give you the decimal expansion of the fraction. Then sort the fractions into two groups: those that are equivalent to terminating decimals and those that are not.

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{2}{6}, \frac{4}{6}, \frac{5}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{11}, \frac{1}{12}, \frac{1}{15}, \frac{1}{16}, \frac{1}{20}, \frac{1}{25}, \frac{1}{32}, \frac{1}{40}, \frac{1}{50}$$

Examine the two groups and answer the following questions:

a. If a unit fraction (a fraction with 1 as the numerator) is represented as a terminating decimal, do all fractions with that denominator terminate?
b. If a unit fraction is represented as a repeating decimal, do all fractions with that denominator repeat?
c. What patterns are observed in the denominators of fractions that represent terminating decimals? That represent repeating decimals?
d. Determine the prime factorization of the denominators of the fractions. Which prime numbers are factors of the denominators of terminating fractions?
e. Why can some fractions be written as terminating decimals while others cannot?

7. Some repeating decimals repeat immediately: for example, the decimal expansion of $\frac{2}{3}$ is $0.\overline{6}$. Other repeating decimal expressions repeat after a delay: for example, the decimal expansion of $\frac{1}{6}$ is $0.1\overline{6}$ (the 1 does not repeat, but the 6 does). What patterns occur in the decimal expansions of repeating decimals? In particular:
   a. Which unit fractions produce decimal expansions that repeat without a delay?
   b. Which unit fractions produce decimal expansions that repeat after a delay?
   c. When converting a unit fraction to a decimal, what is the longest possible period?

8. A student says the fraction 42/150 should be a repeating decimal because the factors of the denominator include a 3 as well as 2s and 5s. But on her calculator 42/150 seems to terminate. How would you explain this?

Operating with Decimals

9. In *Bits and Pieces II,* work through problem 6.2 and the follow-up.

10. Decimals are just fractions whose denominators are powers of 10. Change these decimals to fractions and add them by finding a common denominator.

    0.6 + 0.783 + 0.29

    In what way(s) is this easier than adding fractions such as: $\frac{2}{7}, \frac{5}{6}, \textit{ and } \frac{3}{4}$?

11. In *Bits and Pieces II,* work through problems 6.3, 6.4, and 6.5.

12. Your objective in this activity is to highlight the effects of multiplying and dividing whole numbers by decimal numbers that are close to one, equal to one half, and close to zero.

    Pick a number and multiply it by 0.9, then divide the same number by 0.9. Next, multiply your original number by 0.5, and divide it by 0.5. Finally, multiply the original number by 0.1, and divide it by 0.1. Pick other numbers and perform the same multiplications and divisions.

    What do you notice about the products and the quotients? Try generalizing your conclusions.

13. Discuss the following questions:
    a. When two numbers less than one are multiplied, why is the product smaller than both of the numbers?
    b. How do we explain the fact that when dividing two decimals that are less than one, the quotient is greater than either decimal?

14. Work through "Mathematical Reflections" on pg. 76 from *Bits and Pieces II*.

Technology

15. Take some time in your groups to go through the base 10 blocks applet at the following website: http://arcytech.org/java/b10blocks/. This applet can be used to explore whole numbers or decimals in base 10 as well as in other bases. Play around with this applet – start in base 10 with decimals and then move into other bases. Then answer the following questions:
    a. How could you use this applet to explore decimals in base 10?
    b. How could you use this applet to explore decimals in other bases?
    c. In working with decimals in other bases, what do you discover? How does this help you in your thinking about decimals in our Arabic number system?

Readings:

Case studies:

16. Read "Zeros Sometimes Make a Difference," (p. 42-43) from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?* How would you proceed with Woody, the student in this case?

17. Read "The Decimal Wall," (p. 30-31) and "Lining up the Decimal" (p. 35-36) from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?* Talk

in your groups about the final questions the teachers' pose at the end of each case. What recommendations would you give to each?

18. Read "Testing Theories," (p. 48-51) from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?* What do you think about the teachers approach to helping students develop an understanding of multiplying decimals in this case? Discuss in your groups what you normally do (or would do) in your own classrooms when introducing or exploring decimal multiplication.

19. Read "Where Do I Go From Here," (p. 92-97) from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?* After reading this case, look back at the examples of student work and discuss in your groups how you would help each individual student. Then discuss what you might do for the entire class. Help the teacher decide where to go from here.

20. Read "Everything I Know About Decimals," (p. 11-20) from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?* At the end of the case (epilogue, p. 20) the teacher talks about her hesitance to not drill her kids on pages of fractions and decimals, given the types of questions on most standardized tests. She asks, "where does all this fit in?" (p. 20). Take some time to think about this question – then respond in writing to this teacher, offering her advice.

21. Read "Point Seven plus Point Four is Point Eleven" (p. 26-27) from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?*

22. Read "Bubbles to Kickball," (p. 114-118) from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?* Have students in your classes had similar misconceptions about division remainders? How have you addressed this misunderstanding? Have you ever felt like this teacher – completely at a loss due to total confusion among students? What did you do?

23. Read "Lining Up The Decimal," (p. 35-36) from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?*

Supplemental readings:

24. Now read "2 is Not the Same as 2.0!" from *Mathematics Teaching in the Middle School* (Vol. 6, No. 1, 2000).

Percents

<u>Content Covered</u>:

- Percent Sense
- Operating with Percents
- Ratio and Proportion

<u>SOLs and NCTM Standards</u>:

Associated SOLs

6.1 The student will identify representations of a given percent and describe orally and in writing the equivalence relationships among fractions, decimals, and percents.

6.2 The student will describe and compare two sets of data, using ratios, and will use appropriate notations, such as *a*/*b, a to b*, and *a:b*.

7.4 The student will

a) solve practical problems using rational numbers (whole numbers, fractions, decimals) and percents; and

a) solve consumer–application problems involving tips, discounts, sales tax, and simple interest.

7.6 The student will use proportions to solve practical problems, which may include scale drawings, that contain rational numbers (whole numbers, fractions, and decimals), and percents.

8.3 The student will solve practical problems involving rational numbers, percents, ratios, and proportions. Problems will be of varying complexities and will involve real-life data, such as finding a discount and discount prices and balancing a checkbook.

Associated NCTM Standards

Grades 6 – 8

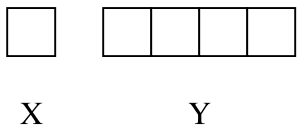
- compare and order fractions, decimals, and percents efficiently and find their approximate locations on a number line
- develop meaning for percents greater than 100 and less than 1
- develop, analyze, and explain methods for solving problems involving proportions, such as scaling and finding equivalent ratios

<u>Problems and Activities (by Content)</u>:

Percent Sense

1. Percent as a Comparison:
   In each item below there are two reference sets, X and Y. Fill in the following statements for each pair of sets: X is ____% of Y, and Y is _____% of X.

a.



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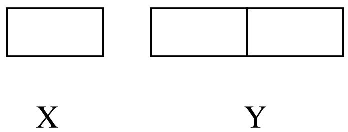
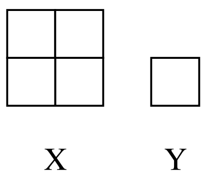
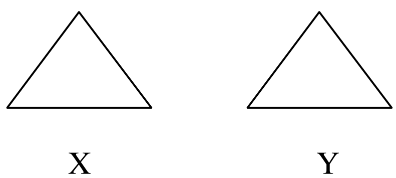


c.



d.



2. While quantities can be represented using decimals, fractions, or percents, which form is used depends a lot on the situation. Come up with several examples of this idea in your groups.

Technology

3. Work through *A Conceptual Model for Solving Percent Problems* (Explorations, pg 79-82).

Operating with Percents

4. In *Bits and Pieces II,* work through the problems in Investigation 1. Discuss how you could use some of these problems in your own classrooms. What do you like about these problems? What would you change about these problems for use in your own classrooms?

5. Solve the following percent problems. Then discuss briefly how you could use each of these problems in your own classroom to generate discussion. What important issues do these problems raise?

   a. In 1990, Mary received a 10% raise. In 1991, due to budget cuts, Mary's boss wanted to cut back salaries to the 1990 rates, so he reduced Mary's salary by 10%. Did Mary's salary go back to the 1990 wage?

   b. Don saw a leather jacket he liked on a 15% off sale. He decided to think about it for few days, and when he went back he noticed the store was giving an additional 10% off. What was the total percentage markdown on the coat?

6. Because of budget cuts, a teacher received a 5% pay cut after her first year of teaching, and another 5% pay cut after her second year on the job. What percent of her starting salary was her salary after 2 years? Explain why the answer is not 90%.

7. Suppose your college bookstore marks up prices 25% from the store’s purchase price. How much did the bookstore actually pay for a textbook with an $85 price tag?

8. What is 0.25% of 345?

9. When Sally was a sophomore, she was able to pay her tuition herself by using the money she earned in the summer. Then there were some tuition hikes. For her junior year, tuition increased by 8%. Then, for her senior year, tuition increased by 8% again. What was the percent increase in tuition from Sally’s sophomore year to her senior year?

10. House prices have risen dramatically over the past 20 years. A house that was purchased in 1980 for $60,000 recently sold for $280,000. What is the percent of increase?

11. Joseph says that if a store has a sale for 35% off and the sale price of a bicycle is $137, then you can figure out what the original price was by taking 35% of $137 and then adding it back onto the $137. So the original price should be $184.95. But that answer doesn’t check. Explain what mistake Jerry is making.

12. It is common practice to pay salespeople extra money called a commission, on the amount of sales. Bill is paid $315 a week, plus 6% commission on sales. Find his total earnings if his sales are $575.

Ratio and Proportion

13. Percent as a Ratio:
    Use equivalent ratios to fill in the blanks in the following statements:
    a. 30% means
       _____ for every 100
       _____ for every 10
       _____ for every 1000
       _____ for every 700
       _____ for every 20
       _____ for every 2
    b. 50% means
       _____ per 100
       _____ per 10
       _____ per 1000
       _____ per 700
       _____ per 20
       _____ per 2
    c. 175% means
       _____ compared with 100
       _____ compared with 10
       _____ compared with 1000
       _____ compared with 700
       _____ compared with 20
       _____ compared with 2
    d. If a sale guarantees a savings of 24%, you will save _____ on every $100, _____ on every $25, _____ on every $50, and _____ on every $250.

14. In *Bits and Pieces II,* work through the problems in Investigation 2.

15. Sue is trying to find the height of a tree in the school yard. She is using the proportion:

$$\frac{\text{Sue's Height}}{\text{Sue's Shadow length}} = \frac{\text{tree's height}}{\text{tree's shadow length}}$$

    Her height is 4 feet. Her shadow length is 15 inches. The length of the tree's shadow is 12 feet. Sue used the proportion $\frac{4}{15} = \frac{tree}{12}$. But that gave the tree's height as being shorter than Sue's! What went wrong?

16. If 80 out of 500 people eat oatmeal for breakfast, what percent of people is this?

<u>Readings</u>:

Case studies:

17. Read "What's My Grade?" (p. 101-102) from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?*

18. Read "How Can 100% of Something Be Just One Thing?" (p. 22-23) from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?*

19. Read "Percents, Proportions, and Grids," (p. 106-109) from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?*

20. Read "A Proportion Puzzle," (p. 37-38) from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?*

21. Read "The Ratio of Girls to Boys," (p. 122) from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?*

22. Read "Knocking Off Zeros," (p. 32-34) from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?*

23. Read "Hugh's Invention," (p. 52-54) from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?*

24. Read "Function Machine," (p. 69-76) from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?*

25. Read "Why Isn't it One Less?" (p. 77-79) from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?*

26. Read "Rote Manipulatives," (p. 82-85) from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?*

27. Read "Favorite-Food Circle Graph" (p. 98-100) from *Fractions, Decimals, Ratios, and Percents: Hard to Teach and Hard to Learn?*

Cryptology

Content Covered:
Cryptology

Problems and Activities:

1) One way to code a message is straight substitution. You can find these and number jumbles in your newspaper, usually near the crossword puzzle. However these codes are easy to crack. We can use various forms of modulo arithmetic to code messages. Decode the Message below (Hint: think Modulo 4!)

   COMQBZAMTRTYSMHHRLEOVZMRBKAUWM
   TNREIDLBCQTDSFIFIJZDSSTMAZUVUWRLN
   QUXIPOCVEINEFLVRAXYSQZKAWUBLUSBA
   ZOWLXNCPQWEHCVBAEIWBSDFEMCUTR

2) Stretch Codes. We can make our codes harder to crack if we use numbers to substitute for letters in our codes. Let's start with the following correspondence:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
P	Q	R	S	T	U	V	W	X	Y	Z				
16	17	18	19	20	21	22	23	24	25	26				

I've included some punctuation. Let's use Base 26 and code a message using a stretch. The message is: NEVER SAY NEVER
First, we change the letters to numbers using the chart above:
14, 5, 22, 5, 18, 19, 1, 25, 14, 5, 22, 5, 18
Next we take each of those numbers and multiply by our stretch factor. Let's use 3.
42, 15, 66, 15, 54, 57, 3, 75, 42, 15, 66, 15, 54
Using Base 26, we next change these to be a number between 1 and 26:
16, 15, 14, 15, 2, 5, 3, 23, 16, 15, 14, 15, 2
Now we convert these numbers back to letters for our code:
PONOB ECW PONOB

Use this coding method to code the following messages in A, B, and C:

A. The Eagle Has Landed
B. You Bet Your Life
C. My Bank Balance is Negative

D. Come up with your own message and decode it using a different stretch factor. Pass this message along to another group to decipher. (Don't tell them your factor!)

3) Using the example in #2, think of a way to reverse the code so that you can take a message coded in this way and decode it to find the original message. Work 4A – C backwards.

4) Decode the following message: Ylvd jvldtfrd bj vfpbdx

5) Another method of coding involves a shift. Again using the table in problem 4 and the message NEVER SAY NEVER, we change our letters to numbers:
14, 5, 22, 5, 18, 19, 1, 25, 14, 5, 22, 5, 18
But this time instead of multiplying, we will add 5 to each number:
19, 10, 27, 10, 23, 24, 6, 30, 19, 10, 27, 10, 23
Again, we adjust the numbers using base 26, so that each are between 1 and 26:
19, 10, 1, 10, 23, 24, 6, 4, 19, 10, 1, 10, 23
This time our coded message is:
SJAJW XFD SJAJW
Use a shift to encode the messages from problem 4
A. The Eagle Has Landed
B. You Bet Your Life
C. My Bank Balance is Negative
D. Come up with your own message and decode it using a different shift factor. Pass this message along to another group to decypher. (Don't tell them your shift factor!)

6) Using the example in #5, think of a way to reverse the code so that you can take a message coded in this way and decode it to find the original message. Work 5A – C backwards.

7) Create your own message and use both a stretch and a shift and modulo 26 to encode your message. Pass this along to another group to decypher.

8) Which of the codes in 2D, 5D and 7 were the easiest to break and why! Explain.

Technology

9) Complete the activities found below the applet on the web page:
http://illuminations.nctm.org/mathlets/codes/index.html

10) Go to the website: http://www.shodor.org/interactivate/lessons/clock.html
Work through the lesson on Caesar Cipher. (You my have already worked through the clock activity earlier in the course) Be sure to complete Caesar Cipher I, Caesar Cipher II and Caesar Cipher III. Compare this applet to the one from the NCTM illuminations website. Which was easier to use? How can these kinds of computer applications be used in the classroom to compliment modulo mathematics?

11) Redo the activities in #2 and 3 above using the applet found at: http://illuminations.nctm.org/mathlets/codes/index.html. Is there any restrictions on what values your stretch can be?

12) Redo the activities in #5 and 6 above using the applet found at: http://illuminations.nctm.org/mathlets/codes/index.html