A SURVEY OF BALANCING METHODS FOR MODEL REDUCTION

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Abstract

Balancing is one of the most efficient and most commonly used methods for model reduction. In this note, we present a survey of several balancing related model reduction schemes. Also we introduce a multiplicative error bound and propose a new reduction method with an absolute error bound for positive real balancing. In addition, a frequency weighted balancing technique with guaranteed stability and a simple bound on the $\mathcal{H}_\infty$ norm of the error system, is presented.

1 Introduction

In this note we consider the linear time invariant dynamical systems in state space form:

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t)+Bu(t) \\ y(t) & = & Cx(t)+Du(t) \end{array} \quad \Leftrightarrow \quad G(s) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \qquad (1)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$. The transfer function of the system (1) is given by $G(s) := C(sI-A)^{-1}B + D$. The problem, we are interested in, is to find a reduced order system $G_r(s) := \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right]$ where $A_r \in \mathbb{R}^{r\times r}$, $B_r \in \mathbb{R}^{r\times m}$, $C_r \in \mathbb{R}^{p\times r}$, $D_r \in \mathbb{R}^{p\times m}$, with $r << n$ such that **(1)** The approximation error is *small*, and there exists a *global* error bound, **(2)** System properties, like *stability*, *passivity*, *boundedness* are preserved and **(3)** The procedure is *computationally stable* and *efficient*.

One of the most efficient and most commonly used reduction techniques is the so-called Lyapunov Balanced Model Reduction first introduced by Mullis and Roberts [14] and later in the systems and control literature by Moore [13]. Balancing is achieved by transforming the system into a basis where the states which are difficult to reach are simultaneously difficult to observe. Then, the reduced model is obtained by truncating the states which have this property. Besides the Lyapunov balancing method, other types of balancing exist, namely *stochastic balancing*, first proposed by Desai and Pal [4] for balancing stochastic systems and later generalized by Green [8, 9]; *bounded real balancing* [16], applied to the bounded-real systems; *positive real balancing* [4], applied for model reduction of positive real (passive) systems; and *frequency weighted balancing* [5, 12, 18, 20, 6] applied to minimize the weighted error over a selected frequency region.

In this note, we present a survey of the balancing related model reduction methods with the corresponding error bounds whenever such a bound exists. In addition, for positive real balancing we introduce a multiplicative error bound and propose a new modified passive reduction technique with an absolute error bound. Also, based on Gawronski and Juang' method [6], a new frequency-weighted balanced reduction method with guaranteed stability and a simple bound on the $\mathcal{H}_\infty$ norm of error system is introduced. A similar work is Ober's paper [15]. However, [15] lacks the error norms and the weighted balancing methods.

The reduced order models via any type of balanced reduction is obtained by truncation of the corresponding balanced basis. Hence we will use the following notation:

$$G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right], G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right] \qquad (2)$$

where $G(s)$ is partitioned in the balanced basis and $G_r(s)$ is the truncated reduced order model with order $r \le n$.

2 Lyapunov Balancing Method

Let $G(s)$ be the to-be-reduced model as defined in (1). Closely related to this system are two continuous time Lyapunov equations $A\mathcal{P}+\mathcal{P}A^T+BB^T=0$ and $A^T\mathcal{Q}+\mathcal{Q}A+C^TC=0$. Under the assumptions that A is asymptotically stable and $G(s)$ is minimal, $\mathcal{P},\mathcal{Q} \in \mathbb{R}^{n\times n}$, called the *controllability* and *observability gramians* respectively, are unique and symmetric positive definite. The square roots of the eigenvalues of the product $\mathcal{PQ}$ are the so-called Hankel singular values $\sigma_i(G(s))$ of the system $G(s)$: $\sigma_i(\mathbf{\Sigma}) = \sqrt{\lambda_i(\mathcal{PQ})}$.

The asymptotically stable and minimal system $G(s)$ is called Lyapunov balanced if $\mathcal{P} = \mathcal{Q} = \Sigma = \mathrm{diag}(\sigma_1 I_{m_1}, \cdots, \sigma_q I_{m_q})$, where $\sigma_1 > \sigma_2 > \cdots > \sigma_q > 0$, m_i, $i = 1, \cdots, q$ are the multiplicities of σ_i, and $m_1 + \cdots + m_q = n$. In this basis, states, which are difficult to reach, are simultaneously difficult to observe and a reduced model is obtained by truncating the states corresponding to small Hankel singular values σ_i.

Theorem 2.1 *Let the asymptotically stable and minimal system $G(s)$ has the Lyapunov balanced realization as in (2) with $\mathcal{P} = \mathcal{Q} = \mathrm{diag}(\Sigma_1, \Sigma_2)$ where $\Sigma_1 = \mathrm{diag}(\sigma_1 I_{m_1}, \cdots, \sigma_k I_{m_k})$ and $\Sigma_2 = \mathrm{diag}(\sigma_{k+1} I_{m_{k+1}}, \cdots, \sigma_q I_{m_q})$. Then the reduced order model $G_r(s)$ in (2) is asymptotically stable, minimal and satisfies*

$$\|G(s) - G_r(s)\|_{\mathcal{H}_\infty} \le 2\,(\sigma_{k+1} + \cdots + \sigma_q). \qquad (3)$$

3 Stochastic Balancing Method

Let $G(s)$ in (1) be asymptotically stable and minimal with **(i)** $m = p$, i.e., $G(s)$ is square, and **(ii)** $\det(D) \neq 0$. Let $W(s)$ be a minimal phase left spectral factor of $G(s)G^{\sim}(s)$, i.e., $W^{\sim}(s)W(s) = G(s)G^{\sim}(s)$ where $G^{\sim}(s) := G^T(-s)$. A realization of $W(s)$ can be computed as $W(s) = \left[\begin{array}{c|c} A & B_W \\ \hline C_W & D^T \end{array}\right]$ with $B_W := \mathcal{P}C^T + BD^T$, and $C_W := D^{-1}(C - B_W^T \mathcal{X})$ where $\mathcal{P}$ is the controllability gramian of $G(s)$, and $\mathcal{X}$ is the solution to the Riccati equation $\mathcal{X}A + A^T\mathcal{X} + \mathcal{X}B_W(DD^T)^{-1}B_W^T\mathcal{X} + C^T(DD^T)^{-1}C = 0$.

The asymptotically stable, minimal, square and non-singular system $G(s)$ is called stochastically balanced if $\mathcal{P} = \mathcal{X} = \operatorname{diag}(\mu_1 I_{t_1}, \cdots, \mu_q I_{t_q})$. where $\mu_1 > \mu_2 > \cdots > \mu_q > 0$, t_i, $i = 1, \cdots, q$ are the multiplicities of μ_i, and $t_1 + \cdots + t_q = n$. μ_i's are indeed the i^{th} Hankel singular value of the stable part of the so-called *phase matrix* $(W^{\sim}(s))^{-1}G(s)$.

Theorem 3.1 *Let the asymptotically stable, minimal, square and non-singular system $G(s)$ be stochastically balanced and partitioned as in (2) with $\mathcal{P}_s = \mathcal{X}_s = \operatorname{diag}(\Gamma_1, \Gamma_2)$ where $\Gamma_1 = \operatorname{diag}(\mu_1 I_{t_1}, \cdots, \mu_k I_{t_k})$ and $\Gamma_2 = \operatorname{diag}(\mu_{k+1} I_{t_{k+1}}, \cdots, \mu_q I_{t_q})$. Then the reduced order model $G_r(s)$ obtained by the stochastic balanced truncation is asymptotically stable, minimal and satisfies*

$$\left\|G^{-1}(G - G_r)\right\|_{\mathcal{H}_\infty} \leq \prod_{i=k+1}^{q} \frac{1+\mu_i}{1-\mu_i} - 1,$$
$$\left\|G_r^{-1}(G - G_r)\right\|_{\mathcal{H}_\infty} \leq \prod_{i=k+1}^{q} \frac{1+\mu_i}{1-\mu_i} - 1$$

In addition, if $G(s)$ is minimum phase, $G_r(s)$ is minimum phase as well.

4 Bounded Real Balancing Method

An important class of the dynamical systems is the so-called bounded real systems whose transfer function is bounded by one on the imaginary axis. This class of systems is used in parameterizing all stabilizing controllers of a system such that the closed-loop system satisfies an $\mathcal{H}_\infty$ constraint [15, 7]. $G(s)$ in (1) is called bounded real if $I - D^T D > 0$ and $I - G^{\sim}(jw)G(jw) > 0, \quad \text{for} \quad \forall w \in \mathbb{R}$.

It can be shown that $G(s)$ is bounded real if and only if there exist a $\mathcal{Y} = \mathcal{Y}^T > 0$ such that

$$\begin{aligned} A^T\mathcal{Y} + \mathcal{Y}A + C^TC + \\ (\mathcal{Y}B + C^TD)R_C^{-1}(\mathcal{Y}B + C^TD)^T = 0, \end{aligned} \tag{4}$$

where $R_C = I - D^TD$. Any solution $\mathcal{Y}$ of (4) lies between two extremal solutions, i.e. $0 < \mathcal{Y}_{\min} \leq \mathcal{Y} \leq \mathcal{Y}_{\max}$. $\mathcal{Y}_{\min}$ is the unique solution to (4) such that $A + BR_C^{-1}(B^T\mathcal{Y} + D^TC)$ is asymptotically stable. Define $R_B := I - DD^T$. Then a dual Riccati equation

$$\begin{aligned} A\mathcal{Z} + \mathcal{Z}A^T + BB^T + \\ (\mathcal{Z}C^T + BD^T)R_B^{-1}(\mathcal{Z}C^T + BD^T)^T = 0. \end{aligned} \tag{5}$$

is obtained where $\mathcal{Z} = \mathcal{Z}^T > 0$. As in the case for (4), any solution $\mathcal{Z}$ of (5) lies between two extremal solutions, i.e. $0 < \mathcal{Z}_{\min} \leq \mathcal{Z} \leq \mathcal{Z}_{\max}$. (4) and (5) are called *the bounded real Riccati equations* of $G(s)$. It is easy to show that if $\mathcal{Y} = \mathcal{Y}^T > 0$ is a solution to (4), then $\mathcal{Z} = \mathcal{Y}^{-1}$ is a solution to (5). Hence $\mathcal{Z}_{\min} = \mathcal{Y}_{\max}^{-1}$ and $\mathcal{Z}_{\max} = \mathcal{Y}_{\min}^{-1}$. Then a bounded real balancing transformation is obtained by balancing $\mathcal{Y}_{\min}$ with $\mathcal{Y}_{\max}^{-1}$, which is equivalent to balancing $\mathcal{Y}_{\min}$ with $\mathcal{Z}_{\min}$.

$G(s)$ is called bounded real balanced if $\mathcal{Y}_{\min} = \mathcal{Z}_{\min} = \mathcal{Y}_{\max}^{-1} = \mathcal{Z}_{\max}^{-1} = \operatorname{diag}(\xi_1 I_{l_1}, \cdots, \xi_q I_{l_q})$ where $1 \geq \xi_1 > \xi_2 > \cdots > \xi_q > 0$, l_i, $i = 1, \cdots, q$ are the multiplicities of ξ_i, and $l_1 + \cdots + l_q = n$. ξ_i's are called *the bounded real singular values* of $G(s)$.

Theorem 4.1 *Let the asymptotically stable, minimal, bounded real system $G(s)$ be bounded real balanced and partitioned as in (2) with $\mathcal{Y}_{\min} = \mathcal{Z}_{\min} = \operatorname{diag}(\Xi_1, \Xi_2)$ where $\Xi_1 = \operatorname{diag}(\xi_1 I_{l_1}, \cdots, \xi_q I_{l_k})$ and $\Xi_2 = \operatorname{diag}(\xi_{k+1} I_{l_{k+1}}, \cdots, \xi_q I_{l_q})$. Let the reduced order model $G_r(s)$ be obtained by truncation as in (2). Then $G_r(s)$ is asymptotically stable, minimal, bounded real balanced and satisfies*

$$\|G(s) - G_r(s)\|_{\mathcal{H}_\infty} \leq 2 \sum_{i=k+1}^{q} \xi_i. \tag{6}$$

5 Positive Real Balancing Method

Another important class of dynamical systems is the so-called positive real (passive) systems. In a physical sense, the positive realness means that the energy produced by the system can never exceed the energy received by it. The asymptotically stable system $G(s)$ in (1) is called positive real if $m = p$, $D^T + D > 0$ and $G^{\sim}(jw) + G(jw) > 0, \quad \text{for} \quad \forall w \in \mathbb{R}$ where $G(s) = C(sI - A)^{-1}B + D$. Define $D_P := D + D^T$. Similarly to the bounded real case, $G(s)$ is positive real if and only if there exist a $\mathcal{K} = \mathcal{K}^T > 0$ such that

$$A^T\mathcal{K} + \mathcal{K}A + (\mathcal{K}B - C^T)D_P^{-1}(\mathcal{K}B - C^T)^T = 0. \tag{7}$$

A dual Riccati equation

$$A\mathcal{L} + \mathcal{L}A^T + (\mathcal{L}C^T - B)D_P^{-1}(\mathcal{L}C^T - B)^T = 0. \tag{8}$$

is also obtained where $\mathcal{L} = \mathcal{L}^T > 0$. (7) and (8) are the so-called *positive real Riccati equations* of $G(s)$. Any solutions $\mathcal{K}$ and $\mathcal{L}$ of, respectively, (7) and (8) lie between two extremal solutions, i.e. $0 < \mathcal{K}_{\min} \leq \mathcal{K} \leq \mathcal{K}_{\max}$ and $0 < \mathcal{L}_{\min} \leq \mathcal{L} \leq \mathcal{L}_{\max}$. If $\mathcal{K} = \mathcal{K}^T > 0$ is a solutions to (7), then $\mathcal{L} = \mathcal{K}^{-1}$ is a solution to (8). Hence $\mathcal{K}_{\min} = \mathcal{L}_{\max}^{-1}$ and $\mathcal{K}_{\max} = \mathcal{L}_{\min}^{-1}$. Then analogously to the bounded real balancing case, a positive real balancing transformation is obtained by balancing the minimal solutions $\mathcal{K}_{\min}$ and $\mathcal{L}_{\min}$ to (7) and (8), respectively.

$G(s)$ is called positive real balanced if $\mathcal{K}_{\min} = \mathcal{L}_{\min} = \mathcal{K}_{\max}^{-1} = \mathcal{L}_{\max}^{-1} = \operatorname{diag}(\pi_1 I_{s_1}, \cdots, \pi_q I_{s_q})$ where $1 \geq \pi_1 > \pi_2 > \cdots > \pi_q > 0$, s_i, $i = 1, \cdots, q$ are the multiplicities of π_i, and $s_1 + \cdots + s_q = n$. π_i are called *the positive real singular values* of $G(s)$.

Denote by $\mathcal{M}$, the Moebius transformation, defined as $\Psi(s) \overset{\mathcal{M}}{\longmapsto} G(s) = (I - \Psi)^{-1}(I + \Psi)$. It is well known that applying a Moebius transformation on a square bounded real

$\Psi(s)$ yields a positive real system. $\mathcal{M}$ is a bijection with inverse $G(s) \overset{\mathcal{M}^{-1}}{\longmapsto} \Psi = (G(s) - I)^{-1}(G(s) + I)^{-1}$, i.e., given a positive real system $G(s)$, $\Psi = \mathcal{M}^{-1}(G(s))$ is bounded real. Then one can show that $\Psi(s)$ is bounded real balanced with bounded real gramians Ξ if and only if $G(s) = \mathcal{M}(\Psi)$ is positive real balanced with positive real gramians $\Pi = \Xi$.

Theorem 5.1 *Let the asymptotically stable, minimal, positive real system $G(s)$ be positive real balanced and partitioned as in (2) with $\mathcal{K}_{\min} = \mathcal{L}_{\min} = \mathrm{diag}(\Pi_1, \Pi_2)$ where $\Pi_1 = \mathrm{diag}(\pi_1 I_{s_1}, \cdots, \pi_k I_{s_k})$ and $\Pi_2 = \mathrm{diag}(\pi_{k+1} I_{s_{k+1}}, \cdots, \pi_q I_{s_q})$. Then $G_r(s)$ obtained by the positive real balanced truncation is asymptotically stable, minimal and positive real balanced.*

Note that there exists no results on the norm of the error $G(s) - G_r(s)$. It is clear that the error results of the stochastic balancing can be employed for positive-real balancing as well. However, in that case *the bounds will be in terms of the spectral factors of $G(s)$, not in terms of $G(s)$*; that is, we will have bounds on the error $\|V^{-1}(V - V_r)\|_\infty$ where $G + G^\sim = V^\sim V$ and $G_r + G_r^\sim = V_r^\sim V_r$. Towards this goal, we state the main result of this section which gives an absolute error bound for $(D^T + G(s))^{-1} - (D^T + G_r(s))^{-1}$ and a multiplicative-like error bound directly in terms of $G(s)$ and $G_r(s)$.

Theorem 5.2 *Given the asymptotically stable positive real system $G(s)$, let $G_r(s)$ be obtained by the positive real balanced truncation as above. Define $R^2 := (D + D^T)^{-1}$, $\alpha_1 := 2\,\|R\|^2 \sum_{i=k+1}^{q} \pi_i$ and $\alpha_2 := \|D^T + G(s)\|_{\mathcal{H}_\infty}$. Then,*

$$\|(D^T + G(s))^{-1} - (D^T + G_r(s))^{-1}\|_{\mathcal{H}_\infty} \leq \alpha_1 \qquad (9)$$

$$\|\ \left(D^T + G_r(s)\right)^{-1}\ (G(s) - G_r(s))\ \|_{\mathcal{H}_\infty} \leq \alpha_1\,\alpha_2 \qquad (10)$$

We state (10) as a multiplicative-like error bound rather than an exact multiplicative error bound because of the term $G_r(s) + D^T$. However, one can easily see that it is a multiplicative error in terms of $G(s) + D^T$ and $G_r(s) + D^T$. One can view (10) also as a weighted error bound where only input weighting exists and is given by $(D^T + G_r(s))^{-1}$.

5.1 A modified positive real balancing method with an absolute error bound

In this section, we will introduce a modified positive real balancing method for a certain subclass of positive real systems. Then based on Theorem 5.2, we will derive an absolute error bound for the proposed method.

Given a positive real system $G(s)$, define $F_G(s)$ as $F_G(s) + R^2 = (G(s) + D^T)^{-1}$. It is clear that $F_G(s) + R^2$ is positive real. To apply the modified algorithm, we will assume that $F_G(s) + R^2/2$ is positive real as well. Therefore, the modified algorithm discussed below will be applicable to the family

$$\mathcal{D} := \left\{G(s) : G(s) \text{ and } F_G(s) + R^2/2 \text{ is positive real}\right\}$$

Through out this section, by positive real systems we mean the positive real systems that belong to the family $\mathcal{D}$. The modified positive real truncation is obtained as follows: Given $G(s) \in \mathcal{D}$, define $H(s)$ with the corresponding D-term D_H as $D_H^T + H(s) = (D^T + G(s))^{-1}$. Since $G \in \mathcal{D}$, $H(s)$ is positive real. Let $\bar{\pi}_i$ denote the positive real singular values of $H(s)$. Then apply the positive real balanced truncation to $H(s)$ to obtain $H_r(s)$ by keeping largest k modified positive real singular values $\bar{\pi}_i$ of $H(s)$. The final reduced order model $\bar{G}_r(s)$ is constructed from $H_r(s)$ using the equality $\bar{D}_r + \bar{G}_r(s) = (D_H^T + H_r(s))^{-1}$. By construction $\bar{D}_r = D$. It follows from the definition of $\mathcal{D}$ that since $H_r(s) \in \mathcal{D}$, $\bar{G}_r(s)$ is positive real as well . Notice that the state-space representations of $H(s)$, $H_r(s)$ and $\bar{G}_r(s)$ are easily obtained through the formulae $D_H^T + H(s) = (D^T + G(s))^{-1}$ and $\bar{D}_r + \bar{G}_r(s) = (D_H^T + H_r(s))^{-1}$.

Theorem 5.3 *Given the positive real system $G(s) \in \mathcal{D}$, let $\bar{G}_r(s)$ be obtained by the modified positive real balancing method introduced above. Then $\bar{G}_r(s)$ is asymptotically stable, positive real and satisfies*

$$\|\ G(s) - \bar{G}_r(s)\ \|_{\mathcal{H}_\infty} \leq 2\,\|R^{-1}\|^2\,(\bar{\pi}_{k+1} + \cdots \bar{\pi}_q). \qquad (11)$$

By Theorem 5.3, we approximate a positive real system $G(s)$ by a reduced order positive real system with an absolute error bound on the $\mathcal{H}_\infty$ norm of the error if $G(s) \in \mathcal{D}$. This error result is analogous to the error result (3) of the Lyapunov balancing and (6) of the bounded real balancing. As Example 7 illustrates if $G(s) \in \mathcal{D}$, the proposed method is a promising alternative to the positive real balancing. The condition $G(s) \in \mathcal{D}$ is still under examination, but we believe, from the experience we gained through a vast amount of numerical examples that, it is not a restrictive condition.

6 Frequency Weighted Balancing Method

All the balancing methods introduced above try to approximate the full order model $G(s)$ over all frequencies. However, in many applications one is only interested in a certain frequency range. This problem leads to the so-called the frequency weighted balancing method. Enns' [5], Lin and Chiu's [12], Wang's *et al.* [18] and Zhou's [20] frequency weighted balanced reduction methods are the most common approaches to attack this problem. These methods construct some input weighting $W_i(s)$ and output weighting $W_o(s)$ and try to minimize the weighted error $\|\ W_o(s)(G(s) - G_r(s))W_i(s)\ \|_{\mathcal{H}_\infty}$. We want to mention that $W_i(s)$ and $W_o(s)$ are some fictitious quantities *unless they are specified by the user*. In most cases, the original problem is to approximate $G(s)$ over a frequency interval $[\ w_1,\ w_2\ ]$ and no input and output weighting is given. Towards this goal, Gawronski and Juang [6] introduced another type of weighted balanced reduction method where for a given frequency band $[\ w_1,\ w_2\ ]$, the construction of the weights are avoided simply by using the frequency domain representation of the gramians. We will propose a frequency weighted balancing method as a modification to [6] which guarantees asymptotic stability and provides a simple error bound.

6.1 Gawronski and Juang's frequency weighted balanced reduction method [6]

In the frequency domain, the controllability gramian $\mathcal{P}$ is given by $\mathcal{P} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(w)BB^T H^*(w)dw$ where $H_w := (jwI - A)^{-1}$. For a given frequency band $\Omega = [\,w_1,\ w_2\,]$ of interest, Gawronski and Juang suggested to choose the frequency weighted controllability gramian as

$$\mathcal{P}_\Omega := \mathcal{P}(w_2) - \mathcal{P}(w_1) \qquad \text{where} \tag{12}$$

$$\mathcal{P}(w) = \frac{1}{2\pi}\int_{-w}^{+w} H(w)BB^T H^*(w)dw. \tag{13}$$

Some tedious manipulations yield that, see [6] for details, $\mathcal{P}(w)$ is the solution to the Lyapunov equation $A\mathcal{P}(w) + \mathcal{P}(w)A^T + W_c(w) = 0$, where $W_c(w) := S(w)BB^T + BB^T S^*(w)$ and $S(w) := \frac{j}{2\pi}\ln((jwI + A)(-jwI + A)^{-1})$. Therefore, the weighted gramian $\mathcal{P}_\Omega$ in (12) is obtained by solving the Lyapunov equation $A\mathcal{P}_\Omega + \mathcal{P}_\Omega A^T + W_c(\Omega) = 0$ where $W_c(\Omega) := W_c(w_2) - W_c(w_1)$. Define $\mathcal{Q}_\Omega$, the weighted observability gramian, analogous to (12) and (13). The similar argument yields $A^T\mathcal{Q}_\Omega + \mathcal{Q}_\Omega A^T + W_o(\Omega) = 0$, where $W_o(\Omega) := W_o(w_2) - W_o(w_1)$, and $W_o(w) := S^*(w)C^TC + C^TCS(w)$. Hence the computations of $\mathcal{P}_\Omega$ and $\mathcal{Q}_\Omega$ require evaluating the logarithm in $S(w)$ in addition to solving two Lyapunov equations. For small-to-medium scale problems for which an exact balancing can be computed, $S(w)$ can be efficiently computed as well.

Gawronski and Juang's frequency weighting method is obtained by balancing $\mathcal{P}_\Omega$ against $\mathcal{Q}_\Omega$, i.e. $\mathcal{P}_\Omega = \mathcal{Q}_\Omega = \operatorname{diag}(\sigma_{n_1}I_{n_1}, \cdots, \sigma_{n_q}I_{n_q})$ where n_i are the multiplicities of each singular value σ_i and $n_1 + \cdots + n_q = n$. Then the reduced order model is obtained by truncation of the balanced basis. However, since $W_c(\Omega)$ and $W_o(\Omega)$ are not guaranteed to be positive definite, stability of the reduced model is never guaranteed.

As seen from the above discussion, the constructions of input and output weightings $W_i(s)$ and $W_o(s)$ are avoided by defining the gramians over the specified frequency range. Indeed one can show that this method is equivalent to Enns's method where $W_i(s)$ and $W_o(s)$ are chosen as the perfect band-pass filters (i.e. an infinite dimensional realization of $W_i(s)$ and $W_o(s)$) over the frequency band $[\,w_1,\ w_2\,]$.

6.2 The modified frequency weighted balanced truncation method

In this section, we will introduce a modification to Gawronski and Juang's and obtain a frequency balancing method which guarantees stability and provides a simple error result.

Let $W_c(\Omega) := M\Lambda M^T = M\operatorname{diag}(\lambda_1, \cdots, \lambda_n)M^T$ and $W_o(\Omega) := N\Delta N^T = N\operatorname{diag}(\delta_1, \cdots, \delta_n)N^T$ be the EVD where $MM^T = NN^T = I_n$ with $|\lambda_1| \geq \cdots |\lambda_n| \geq 0$ and $|\delta_1| \geq \cdots |\delta_n| \geq 0$. Let $\rho := \operatorname{rank}(W_c(\Omega))$ and $\varrho = \operatorname{rank}(W_o(\Omega))$. Inspired by Wang's *et al.* [18] approach, define $\hat{B} := M\operatorname{diag}(|\lambda_1|^{1/2}, \cdots, |\lambda_\rho|^{1/2}, \cdots, 0, \cdots, 0)$ and $\hat{C} := \operatorname{diag}(|\delta_1|^{1/2}, \cdots, |\delta_\varrho|^{1/2}, \cdots, 0, \cdots, 0)N^T$. Then our modified frequency weighted gramians $\bar{\mathcal{P}}_\Omega$ and $\bar{\mathcal{Q}}_\Omega$ are obtained as the solutions to $A\bar{\mathcal{P}}_\Omega + \bar{\mathcal{P}}_\Omega A^T + \hat{B}\hat{B}^T = 0$ and $\bar{\mathcal{Q}}_\Omega A + A^T\bar{\mathcal{Q}}_\Omega + \hat{C}^T\hat{C} = 0$; consequently the modified frequency weighted balancing is obtained by diagonalizing $\bar{\mathcal{P}}_\Omega$ and $\bar{\mathcal{Q}}_\Omega$, i.e. $\bar{\mathcal{P}}_\Omega = \bar{\mathcal{Q}}_\Omega = \operatorname{diag}(\bar{\sigma}_{\tau_1}I_{\tau_1}, \cdots, \bar{\sigma}_{\tau_q}I_{\tau_q})$ where $\bar{\sigma}_i$ are the modified singular values, τ_i are the multiplicities of $\bar{\sigma}_i$ and $\tau_1 + \cdots + \tau_q = n$.

Theorem 6.1 *Let the asymptotically stable system $G(s)$ be in the modified frequency balanced basis as discussed above, and also let G_r be obtained by the truncation of this balanced basis. Assume that* $\operatorname{rank}([\ B\ \ \hat{B}\ ]) = \operatorname{rank}(\hat{B})$ and $\operatorname{rank}([\ C^T\ \ \hat{C}^T\ ]) = \operatorname{rank}(\hat{C}^T)$ *. Then $G_r(s)$ is asymptotically stable, minimal and satisfies*

$$\|\ G(s) - G_r(s)\ \|_{\mathcal{H}_\infty} \leq 2\|J_B\|\|J_C\|(\bar{\sigma}_{k+1} + \cdots + \bar{\sigma}_q)$$

where $J_B := \operatorname{diag}(|\lambda_1|^{-1/2}, \cdots, |\lambda_\rho|^{-1/2}, 0, \cdots, 0)M^TB$ and $J_C := CN\operatorname{diag}(|\delta_1|^{-1/2}, \cdots, |\delta_\varrho|^{-1/2}, 0, \cdots, 0)$.

Note that the assumption $\operatorname{rank}([\ B\ \ \hat{B}\ ]) = \operatorname{rank}(\hat{B})$ is also made in Wang's *et al.* [18] approach. Define $\mathcal{G}(Z) := BZ + Z^TB^T$. Let $\mathcal{G}(Z) = M\Lambda M^T$ be the EVD of $\mathcal{G}(Z)$. Denote $\hat{B} = M\ |\ \Lambda\ |^{1/2}$. It was shown in [18] that for almost all $Z \in \mathbb{C}^{r_1\times n}$, $\operatorname{rank}([\ B\ \ \hat{B}\ ]) = \operatorname{rank}(\hat{B})$. Notice that we have the exact setup with $Z = B^T(S(w_2) - S(w_1))^*$. Hence we expect that our approach will apply in most cases. Indeed, for a vast amount of simulations, the assumption has always been satisfied and it does not seem to be a difficulty in practice..

7 An example on positive-real balancing

Consider a circuit, $G(s)$ consisting of 50 sections interconnected in cascade; each section is as shown in Figure 1-a. The input is the voltage V and the output is the current I, of the first section. The order of $G(s)$ is then $n = 100$. We apply 3 methods, namely **(i)** Positive real balanced reduction (**PRBR**) **(ii)** Modified positive real balanced reduction (**MPRBR**) and **(iii)** Lyapunov balanced reduction (**LBR**); and reduce the order to $r = 10$. We note that $G(s) \in \mathcal{D}$, hence allowing the usage of **MPRBR**. The largest 40 of the normalized Hankel singular values σ_i, the positive real singular values π_i and the modified positive real singular values $\bar{\pi}_i$ of $G(s)$ are shown in Figure 1-b. As the figure illustrates, they all show a very similar decay behavior. This means that for positive real systems, π_i and $\bar{\pi}_i$ play the role of the Hankel singular values. Hence each of these 3 sets of singular values reveal that the decay rate is fast, consequently $G(s)$ is easy to approximate.

The sigma plots of the reduced and error systems are depicted in Figure 1-c and 2-a respectively. Let $G_b(s)$, $G_p(s)$ and $G_m(s)$ denote the reduced models obtained by, respectively, **LBR**, **PRBR** and **MPRBR**. $G_m(s)$ and $G_b(s)$ are very close and slightly better than $G_p(s)$. All the error norms and corresponding upper bounds are tabulated in Table 1. The proposed multiplicative error bound (10) for $G_p(s) + D$ and the absolute error bound (11) for $G_m(s)$ are tight like the upper bound (3) for $G_b(s)$. These results indicate that when $G(s) \in \mathcal{D}$, **MPRBR** is a promising alternative to both **PRBR** and **LBR**.

(a)

(b)

(c)

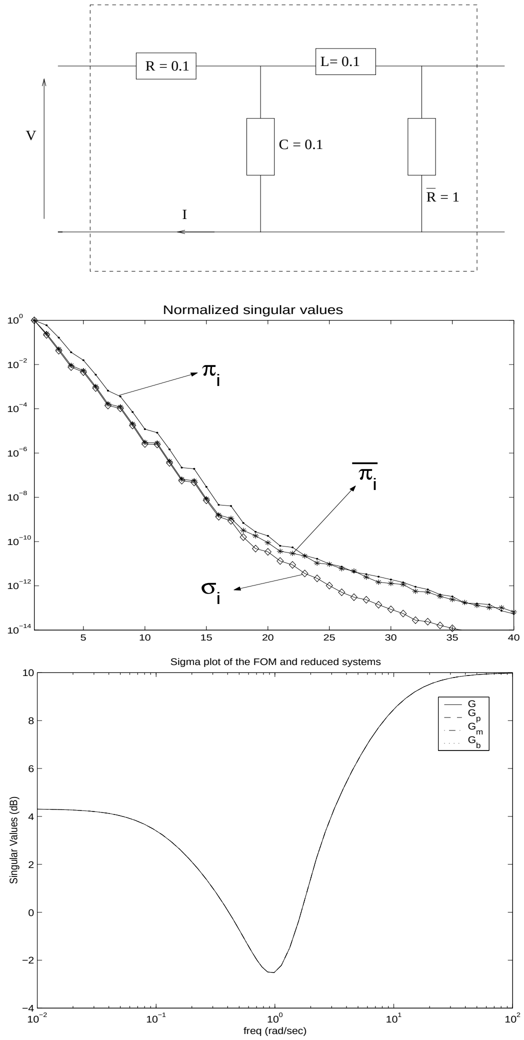

Figure 1: Circuit Example: (a) One section of the circuit (b) σ_i, π_i and $\bar{\pi}_i$ (c) $\sigma_{\max}$ plot of the FOM and ROM

8 An example on frequency weighted balancing

The 120^{th} order SISO full order model (FOM) describes the dynamics of a portable CD player. The sigma plot of the FOM is shown in Figure 2-b. To match the maximum peak of the sigma plot, first we choose $w_1 = 10$ and $w_2 = 1 \times 10^3$. We reduce the order to $r = 15$ by applying **(i)** Gawronski and Juang's method (**GFBT**), **(ii)** our modified frequency balancing method (**MGFBT**) and **(iii)** (unweighted) Lyapunov balanced truncation (**LBT**); and, respectively, obtain the reduced models **(i)** $G_f(s)$, **(ii)** $G_{mf}(s)$ and **(iii)** $G_b(s)$. The sigma plots of the reduced and error systems are depicted in Figure 2-b and Figure 2-c respectively. As Figure 2-c shows $G_f(s)$ and $G_{mf}(s)$ outperform $G_b(s)$ in the chosen frequency interval. Furthermore, $G_{mf}(s)$ and $G(s)$ behaves very similarly. Hence, for this example, our modification to **GFBT** did not have a negative impact on the quality of approximant in the specified region, on the contrary it added the asymptotic stability and an absolute error bound. The $\mathcal{H}_\infty$ errors and corresponding error norms are tabulated in Table 2-a. Now we choose $w_1 = 5 \times 10^3$ and $w_2 = 1 \times 10^5$ to match the ripple in this interval. The sigma

	Exact error	Upper bound
$\|G - G_b\|_{\mathcal{H}_\infty}$	2.7×10^{-5}	2.9×10^{-5}
$\|G - G_m\|_{\mathcal{H}_\infty}$	3×10^{-5}	3.5×10^{-5}
$\|G - G_p\|_{\mathcal{H}_\infty}$	7.4×10^{-5}	
$\|G_{pD}(G - G_p)\|_{\mathcal{H}_\infty}$	5.9×10^{-6}	1.4×10^{-5}
$\|G_D - G_{pD}\|_{\mathcal{H}_\infty}$	$4.6. \times 10^{-7}$	7.2×10^{-7}

Table 1: Error norms and bounds for the Circuit Example where $G_D(s) = (D^T + G(s))^{-1}$ and $G_{pD}(s) = (D^T + G_p(s))^{-1}$

plots are shown in Figures 3-a and 3-b. As expected, $G_f(s)$ and $G_{mf}(s)$ match $G(s)$ and outperforms $G_b(s)$ in the specified interval . On the other hand, in the interval $[\ w_1,\ \ w_2\ ]$, even though $G_{mf}(s)$ matches $G(s)$ quite well, $G_f(s)$ behaves better than $G_{mf}(s)$. This is because of the fact that the modified gramians are no longer the exact frequency-limited gramians, but are close to them. Hence $G_{mf}(s)$ performs slightly worse than $G_f(s)$ over $[\ w_1,\ w_2\ ]$. However the over all response is much better. Notice that while $G_{mf}(s)$ matches the peak of the sigma plot over $[\ 10, 10^3\ ]$ rad/sec, $G_f(s)$ is far from $G(s)$ over this range. The conclusion is that there is a trade-off between the guaranteed stability and the performance in the specified frequency interval. This is also valid for Wang's *et al.* modification to Enns' methods to guarantee stability [18]. The $\mathcal{H}_\infty$ norms and the upper bounds are presented in Table 2-b. The upper bound for $\|G(s) - G_{mf}(s)\|_{\mathcal{H}_\infty}$ is pessimistic because of the fact although **MFBT** is a frequency weighted method, the upper bound is an $\mathcal{H}_\infty$ bound for the whole frequency range.

		Exact error	Upper bound
(a)	$\|G(s) - G_b(s)\|_{\mathcal{H}_\infty}$	4.23×10^{-2}	2.36×10^{-1}
	$\|G(s) - G_{mf}(s)\|_{\mathcal{H}_\infty}$	3.84×10^{-2}	3.40×10^{-1}
	$\|G(s) - G_f(s)\|_{\mathcal{H}_\infty}$	3.85×10^{-2}	
		Exact error	Upper bound
(b)	$\|G(s) - G_b(s)\|_{\mathcal{H}_\infty}$	4.23×10^{-2}	2.36×10^{-1}
	$\|G(s) - G_{mf}(s)\|_{\mathcal{H}_\infty}$	1.45×10^{0}	1.70×10^{1}
	$\|G(s) - G_f(s)\|_{\mathcal{H}_\infty}$	6.83×10^{1}	

Table 2: Error norms for the CD Player Example for (a) $w_1 = 10$ and $w_2 = 1 \times 10^3$ and (b) $w_1 = 5 \times 10^3$ and $w_2 = 1 \times 10^5$

9 Conclusions

We have presented a survey of balancing related model reduction schemes and their corresponding error norms. Two new methods are proposed for positive real and frequency weighted balancing. Moreover, a multiplicative error bound has been introduced for positive real balancing. Two numerical examples have been illustrated to verify the efficiency of the proposed algorithms.

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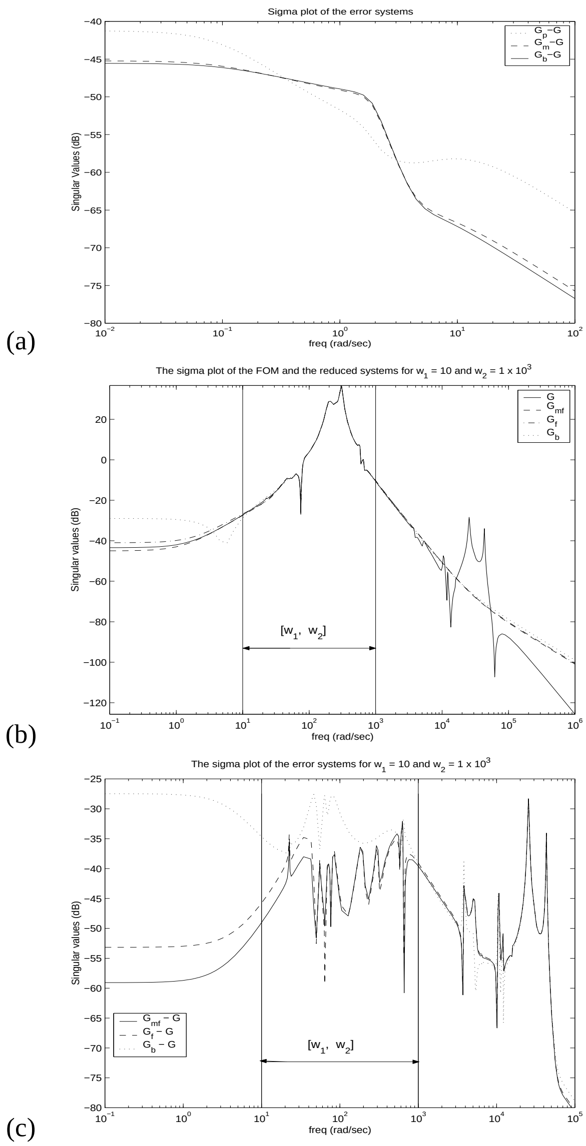

Figure 2: (a) σ_{max} plot of the error systems of the circuit example (b) σ_{max} plot of the reduced and (c) σ_{max} plot of the error systems of the CD player example for $w_1 = 10$ and $w_2 = 1 \times 10^3$

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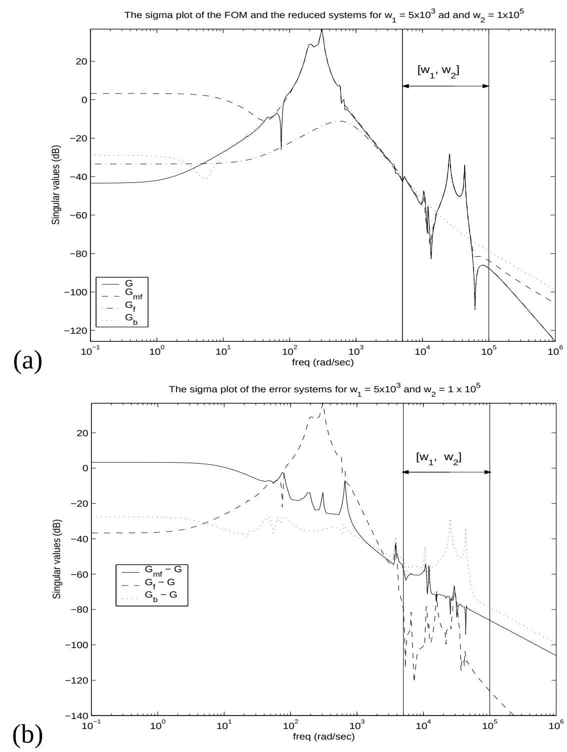

Figure 3: σ_{max} plot of the (a) reduced and (b) error systems of the CD player example for $w_1 = 5 \times 10^3$ and $w_2 = 1 \times 10^5$

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