Proceedings of the 38th
Conference on Decision & Control
Phoenix, Arizona USA • December 1999

On the assignment of eigenvalues in LTI systems

A.C. Antoulas and S. Gugercin
Department of Electrical and Computer Engineering
Rice University, Houston, TX 77005-1892
aca@rice.edu - serkan@rice.edu

Abstract

In this note we discuss the assignability of natural frequencies of LTI systems, given their Hankel singular values. For details we refer to [1].

Preliminaries. Consider the LTI system Σ: $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$, $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^n$, We will assume that Σ is controllable, observable, and stable. There are *two* important invariants associated with Σ: the *natural frequencies or poles* $\lambda_i(\Sigma)$, and the *Hankel singular values* $\sigma_i(\Sigma)$. The former quantities are defined as the eigenvalues of A: $\lambda_i(\Sigma) = \lambda_i(A)$, $i = 1, \cdots, n$. The latter are defined as the singular values of the *Hankel operator* $\mathcal{H}_\Sigma$, associated with Σ: $\mathcal{H}_\Sigma$: $u_- \mapsto y_+ = \mathcal{H}_-(u_-)$, where $y_+(t) := \int_{-\infty}^{0} h(t-\tau)u(\tau)d\tau$, $t > 0$, and $h(t) = Ce^{At}B$, $t \geq 0$, is the impulse response of Σ. It turns out that $\mathcal{H}_\Sigma$ has a finite number of non-zero singular values: $\sigma_i(\Sigma) = \sqrt{\lambda_i(\mathcal{P}\mathcal{Q})}$, $i = 1, \cdots, n$, where $\sigma_1 \geq \cdots \geq \sigma_n > 0$, and $\mathcal{P} > 0$, $\mathcal{Q} > 0$ are the *controllability, observability grammians* of Σ, respectively, satisfying the Lyapunov equations:

$$A\mathcal{P} + \mathcal{P}A^T + BB^T = 0, \quad A^T\mathcal{Q} + \mathcal{Q}A + C^TC = 0$$

The eigenvalues $\lambda_i(\Sigma)$ describe the dynamics of the system. The Hankel singular values $\sigma_i(\Sigma)$ on the other hand - just as the singular values in the case of constant matrices - describe how well a linear dynamical system can be approximated by a similar system of lower dimension.

The **problem** which arises is the *relationship* between $\lambda_i(\Sigma)$ and $\sigma_i(\Sigma)$. More specifically, given the former, to what extent can one influence the latter, and vice versa. We will actually address a more specific question, namely: given $\lambda_i(\Sigma)$, to what extent is it possible to reduce the condition number of the singular values, and given $\sigma_i(\Sigma)$, the condition number of A.

The former problem can be solved by noting that one can always construct an all-pass system with given poles (an all-pass system is perfectly conditioned, since all its Hankel singular values are equal).

Concerning the distribution of the eigenvalues of A for *pre-assigned* Hankel singular values, we first note that these eigenvalues must lie in conic regions which are symmetric with respect to the real axis, with vetrex at zero. These conic regions are bigger, the larger the number of sign changes exhibited by the sequence s_i, $i = 1, \cdots, n$, which is defined below. The issue of the condition number of A can be addressed using the balanced canonical form.

A system representation A, B, C is balanced if the solution of both Lyapunov equations is equal and diagonal, i.e. $\mathcal{P} = \mathcal{Q} = \Sigma$. Every system Σ which is stable, controllable and observable has a balanced representation. Assuming distinct σ_i, the matrices in this case have the following form:

$$B_i = \beta_i > 0,\ C_i = s_i\beta_i,\ A_{ij} = \frac{-\beta_i\beta_j}{s_is_j\sigma_i + \sigma_j}$$

where $s_i = \pm 1$, $i = 1, \cdots, n$. The s_i are signs associated with the singular values σ_i; the quantities $s_i\sigma_i =: \lambda_i$, $i = 1, \cdots, n$, turn out to be the eigenvalues of the Hankel operator $\mathcal{H}_\Sigma$. From the above relationships it follows that A can be written as $A = B_dA_0B_d$, $(A_0)_{ij} = \frac{-1}{s_is_j\sigma_i+\sigma_j}$, $B_d = \text{diag}\,(\beta_1, \cdots, \beta_n)$, $\beta_i, \sigma_i > 0$.

Theorem 1 *Given n distinct positive real numbers σ_i, together with n signs s_i, the condition number of $A = B_dA_0B_d$ is minimized, for the following choice of the entries of B_d:*

$$\beta_i^2 = |\lambda_i + \lambda_i| \prod_{j\neq i} \left| \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right|$$

where $\lambda_k = s_k\sigma_k$, $k = 1, \cdots, n$.

Remark. Matrices of the type defined by A_0 are *Cauchy* matrices, while for the special case that all signs s_i are positive we have *Hilbert matrices*. Thus the theorem above provides the optimal diagonal scaling of Cauchy and Hilbert matrices.

References

[1] A.C. Antoulas, *On eigenvalues and singular values of linear dynamical systems*, Proc. XIV Housholder Symposium on Numerical Linear Algebra, June 1999.