Inexact Solves in Krylov-based Model Reduction of Large-scale Dynamical Systems

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Introduction

- Consider an n^{th} order single-input/single-output system $\mathbf{H}(s)$:

$$\mathbf{H}(s) : \begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{c}\,\mathbf{x}(t) \end{cases} \quad \Leftrightarrow \quad \begin{aligned} \mathbf{H}(s) &= \mathbf{c}(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{b} \\ &= \frac{\mathbf{p}_{n-1}(s)}{\mathbf{q}_n(s)} \end{aligned}$$

- $\mathbf{u}(t) \in \mathbb{R}$: input, $\quad \mathbf{x}(t) \in \mathbb{R}^n$: state, $\quad \mathbf{y}(t) \in \mathbb{R}$: output

- $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\quad \mathbf{b}, \mathbf{c}^T \in \mathbb{R}^n$. Will assume $\Re(\lambda_i(\mathbf{A})) < 0$

- Need for improved accuracy $\Longrightarrow$ Include more details in the modeling stage

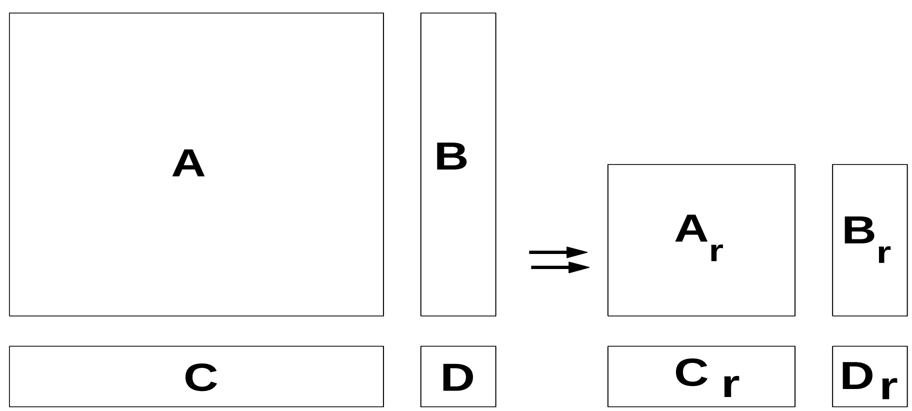
- In many applications, n is quite large, $n \approx \mathcal{O}(10^6, 10^7)$,

- Untenable demands on computational resources $\Longrightarrow$

Model Reduction Problem: Find

$$\begin{aligned} \dot{\mathbf{x}}_r(t) &= \mathbf{A}_r\,\mathbf{x}_r(t) + \mathbf{b}_r\,\mathbf{u}(t) \\ \mathbf{y}_r(t) &= \mathbf{c}_r\,\mathbf{x}_r(t) \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \mathbf{H}_r(s) &= \mathbf{c}_r(s\mathbf{I}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r \\ &= \frac{\mathbf{p}_{r-1}(s)}{\mathbf{q}_r(s)} \end{aligned}$$

- where $\mathbf{A}_r \in \mathbb{R}^{r\times r}$, $\quad \mathbf{b}_r, \mathbf{c}_r^T \in \mathbb{R}^r$, with $r \ll n$ such that
  1. $\|\mathbf{y} - \mathbf{y}_r\|$ is *small.*
  2. The procedure is *computationally efficient.*



- $\mathbf{G}_r(s)$: used for simulation or designing a reduced-order controller

- Model reduction through projection: a unifying framework.
- Construct $\mathbf{\Pi} = \mathbf{V}\mathbf{W}^T$, where $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n\times r}$ with $\mathbf{W}^T\mathbf{V} = \mathbf{I}_r$:

$$\dot{\mathbf{x}}_r = \underbrace{\mathbf{W}^T\mathbf{A}\mathbf{V}}_{:=\mathbf{A}_r}\mathbf{x}_r(t) + \underbrace{\mathbf{W}^T\mathbf{b}}_{:=\mathbf{b}_r}\mathbf{u}(t), \quad \mathbf{y}_r(t) = \underbrace{\mathbf{c}\mathbf{V}}_{:=\mathbf{c}_r}\mathbf{x}_r(t)$$

What is the approximation error $\mathbf{e}(t) := \mathbf{y}(t) - \mathbf{y}_r(t)$?

- $\mathbf{G}(s)$: Associate a convolution operator $\mathcal{S}$:

$$\mathcal{S} : \mathbf{u}(t) \mapsto \mathbf{y}(t) = (\mathcal{S}\mathbf{u})(t) = (\mathbf{g} \star \mathbf{u})(t) = \int_{-\infty}^{t} \mathbf{g}(t-\tau)\mathbf{u}(\tau)d\tau.$$

- $\mathbf{h}(t) = \mathbf{c}e^{\mathbf{A}t}\mathbf{b}$ for $t \ \geq 0$: *Impulse response*.
- Transfer function: $\mathbf{H}(s) = (\mathcal{L}\mathbf{h})(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$.

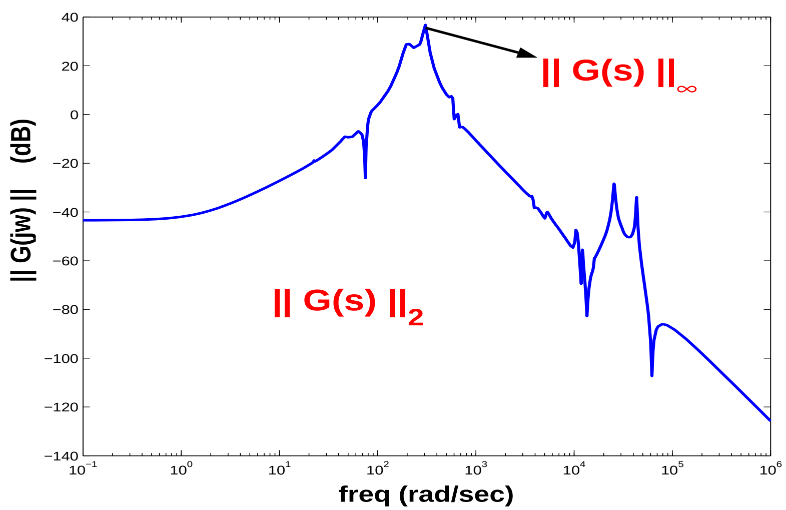
The $\mathcal{H}_\infty$ Norm : 2-2 induced norm of $\mathcal{S}$:

$$\|\mathbf{H}(s)\|_{\mathcal{H}_\infty} = \sup_{\mathbf{u}\neq 0} \frac{\|\mathbf{y}\|_2}{\|\mathbf{u}\|_2} = \sup_{\mathbf{u}\neq 0} \frac{\|\mathcal{S}u\|_2}{\|\mathbf{u}\|_2} = \sup_{w\in\mathbb{R}} \|\mathbf{H}(\jmath w)\|_2$$

$$\|\mathbf{H} - \mathbf{H}_r\|_\infty = \text{Worst output error } \|\mathbf{y}(t) - \mathbf{y}_r(t)\|_2 \quad \forall \quad \|\mathbf{u}(t)\|_2 = 1.$$

The $\mathcal{H}_2$ Norm : $\mathcal{L}_2$ norm of $\mathbf{g}(t)$ in time domain:

$$\|\mathbf{H}(s)\|^2_{\mathcal{H}_2} = \int_0^\infty \text{trace}[\mathbf{h}^T(t)\mathbf{h}(t)]dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \text{trace}[\mathbf{H}^*(\jmath w)\mathbf{H}(\jmath w)]dw$$



Model Reduction via Interpolation

Rational Interpolation: Given $\mathbf{H}(s)$, find $\mathbf{H}_r(s)$ so that

$\mathbf{H}_r(s)$ *interpolates* $\mathbf{H}(s)$ and certain number of its derivatives at selected frequencies σ_k in the complex plane

$$\frac{(-1)^j}{j!}\frac{d^j\mathbf{H}(s)}{ds^j}\bigg|_{s=\sigma_k} = \frac{(-1)^j}{j!}\frac{d^j\mathbf{H}_r(s)}{ds^j}\bigg|_{s=\sigma_k}, \quad \begin{array}{l}\text{for } k=1,\ldots,K,\\ \text{and } j=1,\ldots,J\end{array}$$

- $$\frac{(-1)^j}{j!}\frac{d^j\mathbf{H}(s)}{ds^j}\bigg|_{s=\sigma_k} = \mathbf{c}(\sigma_k\mathbf{I}-\mathbf{A})^{-(j+1)}\mathbf{b}:$$
  $= j^{\text{th}}$ **moment** of $\mathbf{H}(s)$ at σ_k
  $=: {\eta_{\sigma_k}}^{(j)}$.

Rational Krylov Method

- Moments are very ill-conditioned to compute
- Given r interpolation points: $\{\sigma_i\}_{i=1}^{r}$
- Set $\mathbf{V} = \text{Span}\left[(\sigma_1\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}, \cdots, (\sigma_r\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}\right]$, and
- $\mathbf{W} = \text{Span}\left[(\overline{\sigma_1}\,\mathbf{I}-\mathbf{A}^T)^{-1}\mathbf{c}^T, \cdots, (\overline{\sigma_r}\,\mathbf{I}-\mathbf{A}^T)^{-1}\mathbf{c}^T\right]$. Make $\mathbf{W}^T\mathbf{V} = \mathbf{I}_r$.
- $\mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}, \ \mathbf{b}_r = \mathbf{W}^T\mathbf{b}, \ \mathbf{c}_r = \mathbf{c}\mathbf{V}$

$$\Longrightarrow \boxed{\mathbf{H}(\sigma_i) = \mathbf{H}_r(\sigma_i), \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{H}(s)\Big|_{s=\sigma_i} = \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{H}_r(s)\Big|_{s=\sigma_i}}$$

- Moment matching without explicit moment computation
- Efficient computations of $\mathbf{V}$ and $\mathbf{W}$: Grimme [1997]

Inexact solves in Krylov-based model reduction

- Need for more detail and accuracy in the modeling stage $\Rightarrow$
- System dimension n: $\mathcal{O}(10^6)$ or more $\Rightarrow$
- $(\sigma\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{b}$ cannot be solved directly
- Inexact solves need to be employed in constructing $\mathbf{V}$ and $\mathbf{W}$
- Questions:
  1. What are the perturbation effects on interpolation?
  2. What are the effective preconditioning, restarting strategies?
  3. What is the effect on (the optimality of) the reduced model?

- For simplicity, consider the one-sided projection, i.e. $\mathbf{V} = \mathbf{W}$.
- Let $\hat{\mathbf{v}}_j$ be an inexact solution for $(\sigma_j \mathbf{I} - \mathbf{A})\mathbf{v}_j = \mathbf{b}$

$$(\sigma_j \mathbf{I} - \mathbf{A})\hat{\mathbf{v}}_j - \mathbf{b} = \delta \mathbf{b}_j$$

- Define $\quad \delta \mathbf{v}_j := \hat{\mathbf{v}}_j - \mathbf{v}_j = (\sigma_j \mathbf{I} - \mathbf{A})^{-1} \delta \mathbf{b}_j$, and

$$\hat{\mathbf{K}} := \left[\ (\sigma_1 \mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + \delta \mathbf{v}_1, \quad \cdots \quad (\sigma_r \mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + \delta \mathbf{v}_r\ \right].$$

- Inexact Krylov-based reduced model obtained by

$$\widehat{\mathbf{A}}_r = \widehat{\mathbf{V}}^T \mathbf{A} \widehat{\mathbf{V}}, \quad \widehat{\mathbf{b}}_r = \widehat{\mathbf{V}}^T \mathbf{b}, \quad \widehat{\mathbf{c}}_r = \mathbf{c} \widehat{\mathbf{V}}, \quad \text{where} \quad \widehat{\mathbf{V}}^T \widehat{\mathbf{V}} = \mathbf{I}_r.$$

- $\widehat{\mathbf{V}}$: an orthogonal basis for Range($\hat{\mathbf{K}}$)
- $\widehat{\mathbf{H}}_r(s) = \widehat{\mathbf{c}}_r(s\mathbf{I}_r - \widehat{\mathbf{A}}_r)^{-1}\widehat{\mathbf{b}}_r$

Theorem: **The response (interpolation) error** at σ_j is

$$\boxed{\varepsilon_j := \widehat{\mathbf{H}}_r(\sigma_j) - \mathbf{H}(\sigma_j) = \|\delta\mathbf{b}_j\| \cdot \mathbf{c}\,\mathbf{M}_j\,\frac{\delta\mathbf{b}_j}{\|\delta\mathbf{b}_j\|}}$$

- $\mathbf{M}_j = \left[(\sigma_j\mathbf{I}_n - \mathbf{A})^{-1} - \widehat{\mathbf{V}}(\sigma_j\mathbf{I}_r - \mathbf{A}_r)^{-1}\widehat{\mathbf{V}}^T\right]$
- How well $\widehat{\mathbf{V}}(\sigma_j\mathbf{I}_r - \mathbf{A}_r)^{-1}\widehat{\mathbf{V}}^T$ approximates $(\sigma_j\mathbf{I}_n - \mathbf{A})^{-1}$

  $\Rightarrow$ Quality of the Ritz approximation $\widehat{\mathbf{V}}\mathbf{A}_r\widehat{\mathbf{V}}^T$ to $\mathbf{A}$.

- Depends on the selection of σ_j: Good selection $\Rightarrow$ Krylov-based model reduction robust with respect to inexact solves.

- **Backward Error:**

$$\boxed{\widehat{\mathbf{H}}_r(\sigma_j) = \mathbf{H}^{[j]}(\sigma_j) = \mathbf{c}(s\mathbf{I}_n - \mathbf{A})^{-1}(\mathbf{b} + \Delta\mathbf{b}_j)}$$

- Exact interpolation of a near-by system.

The two-sided case

- $\hat{\mathbf{K}}_1 := \left[\ (\sigma_1\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + \delta\mathbf{v}_1, \quad \cdots \quad (\sigma_r\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + \delta\mathbf{v}_r\ \right].$
- Let $\hat{\mathbf{w}}_j$ be an inexact solution for $(\overline{\sigma_j}\mathbf{I} - \mathbf{A}^T)\mathbf{w}_j = \mathbf{c}^T$

$$(\overline{\sigma_j}\mathbf{I} - \mathbf{A}^T)\hat{\mathbf{w}}_j - \mathbf{c}^T = \delta\mathbf{c}_j$$

- Define $\qquad \delta\mathbf{w}_j := \hat{\mathbf{w}}_j - \mathbf{w}_j = (\overline{\sigma_j}\mathbf{I} - \mathbf{A})^{-1}\delta\mathbf{c}_j$, and

  $\hat{\mathbf{K}}_2 := \left[\ (\overline{\sigma_1}\mathbf{I} - \mathbf{A}^T)^{-1}\mathbf{c}^T + \delta\mathbf{w}_1, \cdots, (\overline{\sigma_r}\mathbf{I} - \mathbf{A}^T)^{-1}\mathbf{c}^T + \delta\mathbf{w}_r\ \right].$

- Inexact Krylov-based reduced model obtained by

$$\widehat{\mathbf{A}}_r = \widehat{\mathbf{W}}^T\mathbf{A}\widehat{\mathbf{V}}, \quad \widehat{\mathbf{b}}_r = \widehat{\mathbf{W}}^T\mathbf{b}, \quad \widehat{\mathbf{c}}_r = \mathbf{c}\widehat{\mathbf{V}}$$

- $\widehat{\mathbf{V}}, \widehat{\mathbf{W}}$: biorthogonal basis for Range($\hat{\mathbf{K}}_1$) and Range($\hat{\mathbf{K}}_2$)
- $\widehat{\mathbf{H}}_r(s) = \widehat{\mathbf{c}}_r(s\mathbf{I}_r - \widehat{\mathbf{A}}_r)^{-1}\widehat{\mathbf{b}}_r$

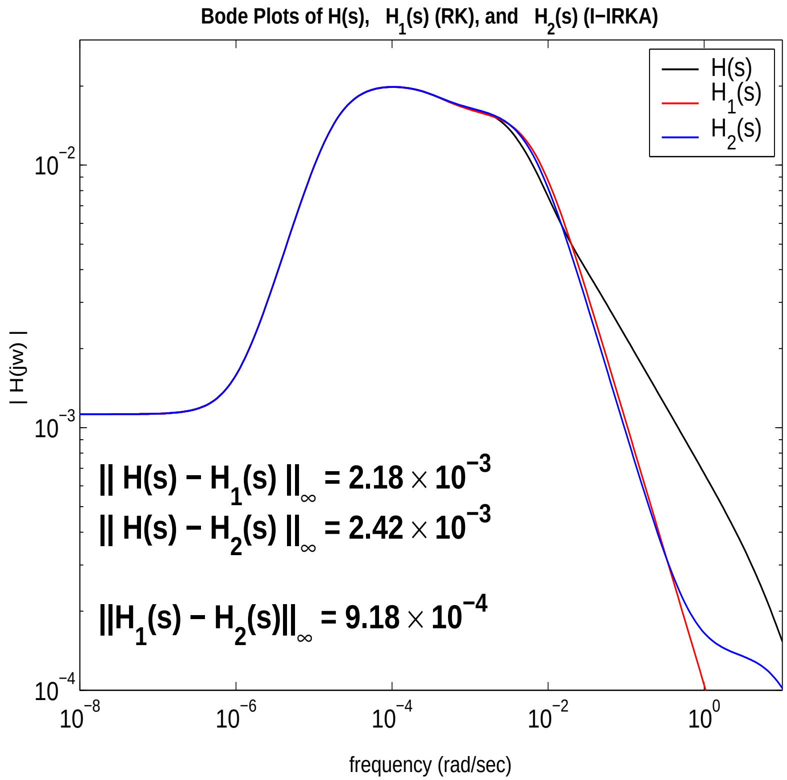
Theorem: The response (interpolation) error at $\{\sigma_j\}_{j=1}^{r}$ is

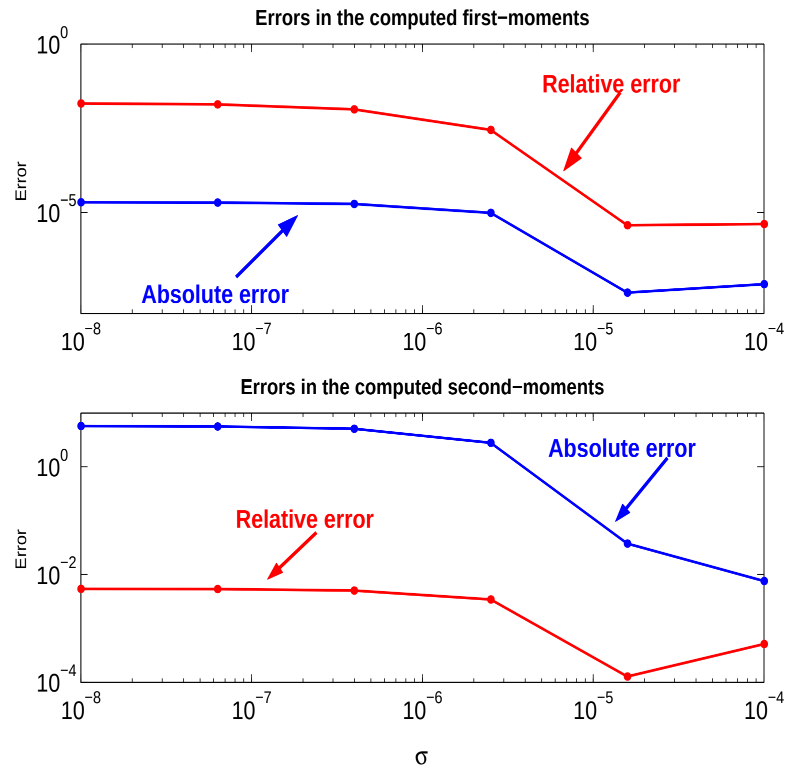
$$\boxed{\varepsilon_j := \widehat{\mathbf{H}}_r(\sigma_j) - \mathbf{H}(\sigma_j) = \|\delta\mathbf{b}_j\| \cdot \|\delta\mathbf{c}_j\| \cdot \frac{\delta\mathbf{c}_j^T}{\|\delta\mathbf{c}_j\|}\mathbf{M}_j(\sigma_j\mathbf{I}_n - \mathbf{A})\mathbf{M}_j\frac{\delta\mathbf{b}_j}{\|\delta\mathbf{b}_j\|}}$$

- $\mathbf{M}_j = \left[(\sigma_j\mathbf{I}_n - \mathbf{A})^{-1} - \widehat{\mathbf{V}}(\sigma_j\mathbf{I}_r - \mathbf{A}_r)^{-1}\widehat{\mathbf{W}}^T\right]$
- How well $\widehat{\mathbf{V}}(\sigma_j\mathbf{I}_r - \mathbf{A}_r)^{-1}\widehat{\mathbf{W}}^T$ approximates $(\sigma_j\mathbf{I}_n - \mathbf{A})^{-1}$
- Quality of the Galerkin approximation $\widehat{\mathbf{V}}\mathbf{A}_r\widehat{\mathbf{W}}^T$ to $\mathbf{A}$.
- Depends on the selection of σ_j
- Quadratic feature of the error
- Similar backward error estimate as the one-sided case

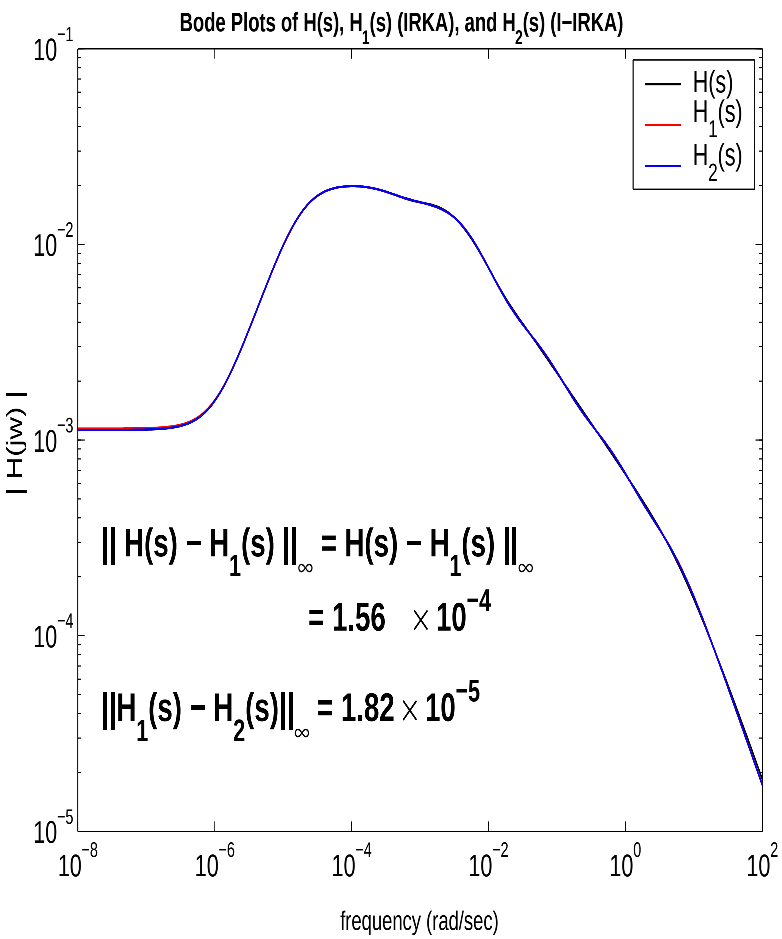
Example: Optimal Cooling of Steel Profiles (P. Benner)

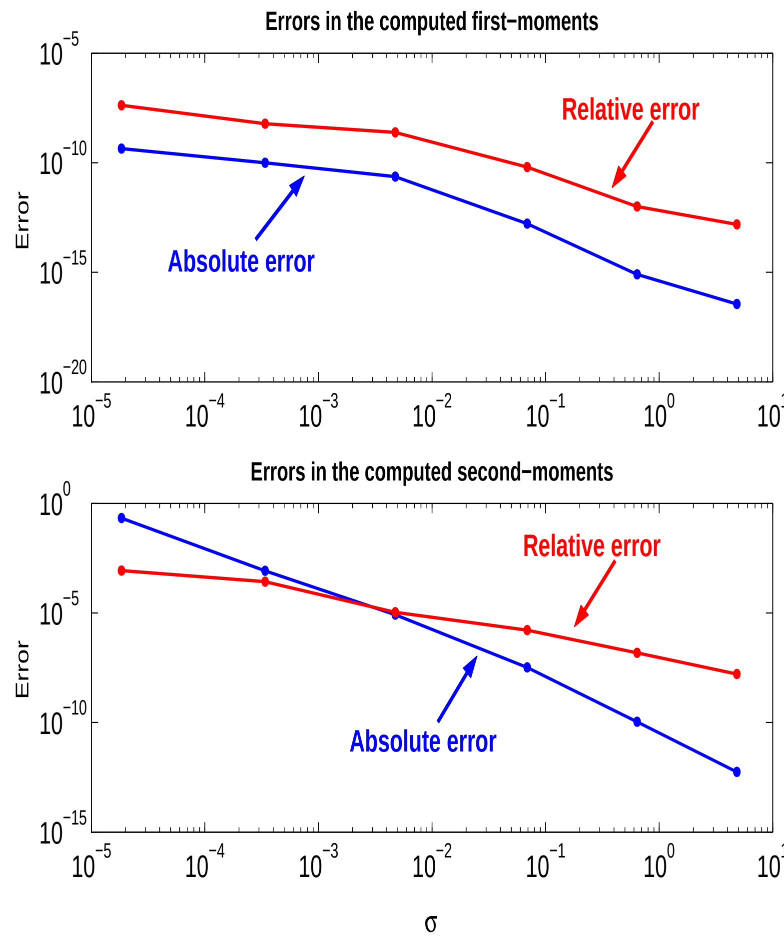
- $\mathbf{H}(s) = \mathbf{c}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$, $n = 20,209$
- Bad shift selection: $\sigma_i = logspace(-8, -4, 6)$
- $r = 6$ via Rational Krylov (**RK**) and Inexact-**RK** (**I-RK**).
- **I-RK** uses GMRES with $tol = 10^{-5}$





- **RK** with optimal $\{\sigma_i\}$
- Use these optimal $\{\sigma_i\}$ in **I-RK**.
- **I-RK** uses GMRES with $tol = 10^{-4}$





- **GMRES**:
  1. The same Krylov subspace for each $(\sigma_j \mathbf{I} - \mathbf{A})\mathbf{v}_j = \mathbf{b}$
     $\mathbf{A}\mathbf{U}_k = \mathbf{U}_{k+1}\widetilde{\mathbf{H}}_k \Rightarrow \min \left\| \sigma_j \tilde{\mathbf{I}} - \widetilde{\mathbf{H}}_k - \|\mathbf{b}\|\mathbf{e}_1 \right\|$
  2. Span$\{\mathbf{v}_j\}_{j=1}^{r}$ is important, rather than each $\mathbf{v}_j$
     $\Longrightarrow$ One could afford a less accurate solution for $\mathbf{v}_j$ if already included in the subspace span $\{\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}\}$

- **Preconditioning**:
  1. If σ_j is *close* to σ_{j+1}, can re-use preconditioners for different linear systems
  2. Cost of recomputing vs cost of using a close-by preconditioner

Optimal $\mathcal{H}_2$ approximation

Problem: Given a stable dynamical system $\mathbf{H}(s)$, find a reduced model $\mathbf{H}_r(s)$ that satisfies

$$\mathbf{H}_r(s) = \arg \min_{\substack{\deg(\hat{\mathbf{H}}) = r \\ \hat{\mathbf{H}}\,:\,\text{stable}}} \left\| \mathbf{H}(s) - \hat{\mathbf{H}}(s) \right\|_{\mathcal{H}_2}.$$

- First-order conditions: (Meier and Luenberger [1967])

$$\mathbf{H}(-\hat{\lambda}_i) = \mathbf{H}_r(-\hat{\lambda}_i), \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{H}(s)\Big|_{s=-\hat{\lambda}_i} = \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{H}_r(s)\Big|_{s=-\hat{\lambda}_i}$$

- Match the first two moments at the mirror images of the Ritz values.
- First-order conditions as interpolation. $\Rightarrow$
- Rational Krylov Framework

- For the $\mathcal{H}_2$ problem, simply set $\boxed{\sigma_i = -\hat{\lambda}_i}$
- $\hat{\lambda}_i$ NOT known a priori $\Longrightarrow$ Needs iterative rational steps

An Iterative Rational Krylov Algorithm (IRKA):
(G./Beattie/Antoulas [2004])

1. Choose σ_i for $i = 1, \ldots, r$.
2. $\mathbf{V} = \text{Span}\left[(\sigma_1\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}, \cdots, (\sigma_r\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\ \right]$,
3. $\mathbf{W} = \text{Span}\left[(\overline{\sigma_1}\,\mathbf{I} - \mathbf{A}^T)^{-1}\mathbf{c}^T, \cdots, (\overline{\sigma_r}\,\mathbf{I} - \mathbf{A}^T)^{-1}\mathbf{c}^T\ \right]$, $\mathbf{W}^T\mathbf{V} = \mathbf{I}_r$.
4. while [relative change in σ_j] $> \epsilon$
   (a) $\mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}$,
   (b) $\sigma_i \longleftarrow -\lambda_i(\mathbf{A}_r)$ for $i = 1, \ldots, r$
   (c) $\mathbf{V} = \text{Span}\left[(\sigma_1\,\mathbf{I} - \mathbf{A})^{-1}\mathbf{b},\ \cdots, (\sigma_r\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\right]$.
   (d) $\mathbf{W} = \text{Span}\left[(\overline{\sigma_1}\mathbf{I} - \mathbf{A}^T)^{-1}\mathbf{c}^T, \cdots, (\overline{\sigma_r}\,\mathbf{I} - \mathbf{A}^T)^{-1}\mathbf{c}^T\ \right]$, $\mathbf{W}^T\mathbf{V} = \mathbf{I}_r$.
5. $\mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}$, $\quad \mathbf{b}_r = \mathbf{W}^T\mathbf{b}$, $\quad \mathbf{c}_r = \mathbf{c}\mathbf{V}$

- Upon convergence, first-order conditions satisfied via Krylov projection framework, no Lyapunov solvers

Inexact IRKA (I-IRKA)

- IRKA requires solving $2r$ linear systems at each step
  $\Rightarrow$ Expensive if $n = \mathcal{O}(10^6)$
- In most cases $\{\sigma_j\}$ converge fast

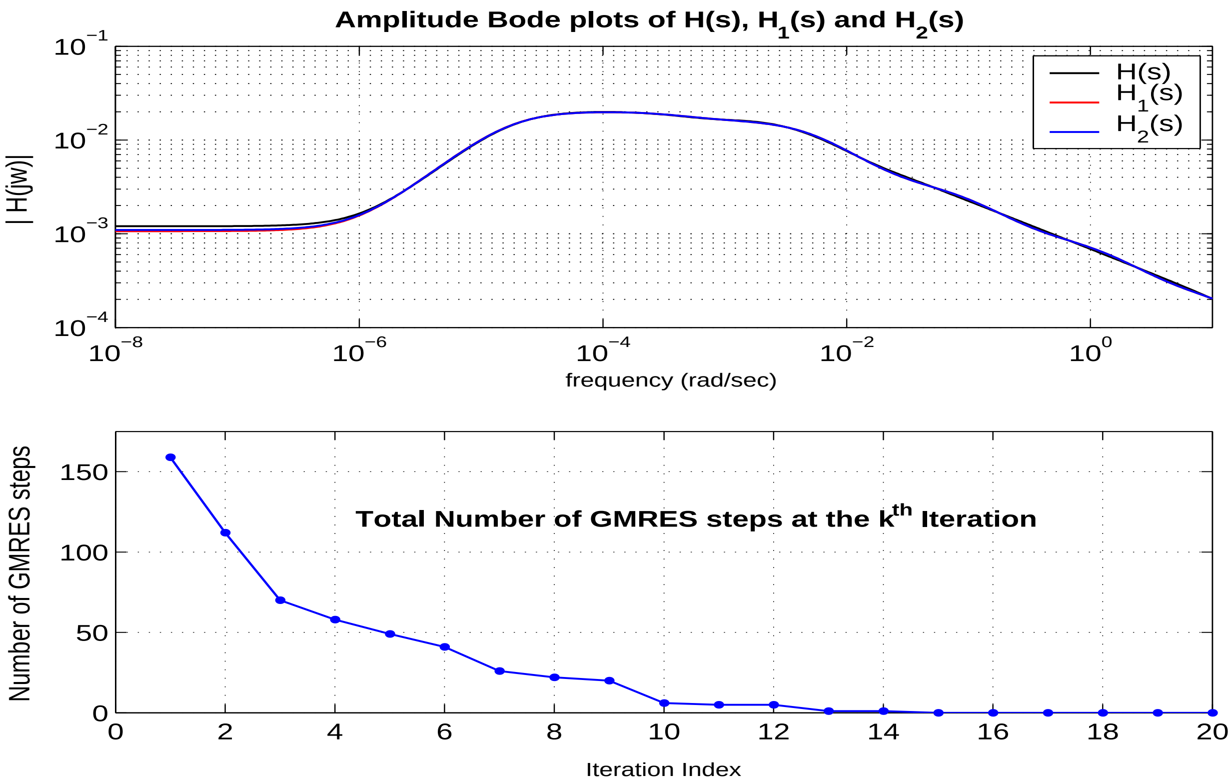
$$\Downarrow$$

- Use the solution from the previous step as an initial guess for the next step
- Expect faster convergence for a fixed tolerance value
- Optimal reduced model: Expect robustness

An Inexact Iterative Rational Krylov Algorithm (I-IRKA):

1. Make an initial shift selection σ_i for $i = 1, \ldots, r$
2. for $i = 1, \ldots, r$
   (a) $\mathbf{v}_i = \mathbf{f}\left(\mathbf{A}, \mathbf{b}, \sigma_i, \mathbf{0}, \epsilon\right)$ ($\mathbf{f}(\mathbf{A}, \mathbf{b}, \sigma, \mathbf{0}, \epsilon)$: an iterative solve for $(\sigma\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{b}$))
   (b) $\mathbf{w}_i = \mathbf{f}\left(\mathbf{A}^T, \mathbf{c}^T, \sigma_i, \mathbf{x}_0, \epsilon\right)$
3. $\mathbf{W} = [\ \mathbf{w}_1,\ \mathbf{w}_2,\ \ldots,\ \mathbf{w}_r\ ]$, $\mathbf{V} = [\ \mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_r\ ]$.
4. $\mathbf{W} = \mathbf{W}(\mathbf{W}^T\mathbf{V})^{-T}$ (to make $\mathbf{W}^T\mathbf{V} = \mathbf{I}_r$)
5. while (not converged)
   (a) $\mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}$,
   (b) $\sigma_i \longleftarrow -\lambda_i(\mathbf{A}_r)$ for $i = 1, \ldots, r$
   (c) for $i = 1, \ldots, r$
      i. $\mathbf{v}_i = \mathbf{f}\left(\mathbf{A}, \mathbf{b}, \sigma_i, \mathbf{v}_i, \epsilon\right)$
      ii. $\mathbf{w}_i = \mathbf{f}\left(\mathbf{A}^T, \mathbf{c}^T, \sigma_i, \mathbf{w}_i, \epsilon\right)$
   (d) $\mathbf{W} = [\ \mathbf{w}_1,\ \mathbf{w}_2,\ \ldots,\ \mathbf{w}_r\ ]$, $\mathbf{V} = [\ \mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_r\ ]$.
   (e) $\mathbf{W} = \mathbf{W}(\mathbf{W}^T\mathbf{V})^{-T}$ (to make $\mathbf{W}^T\mathbf{V} = \mathbf{I}_r$)
6. $\mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}$, $\mathbf{b}_r = \mathbf{W}^T\mathbf{b}$, $\mathbf{c}_r = \mathbf{c}\mathbf{V}$

- Same model with $n = 79,841$ (Finer discretization)
- $r = 6$ via **IRKA** and **I − IRKA** ($tol = 5 \times 10^{-5}$)
- **IRKA**: Initial guess from the previous step



- $\|\mathbf{H}(s) - \mathbf{H}_1(s)\|_\infty = \|\mathbf{H}(s) - \mathbf{H}_2(s)\|_\infty = 6.01 \times 10^{-5}$, $\|\mathbf{H}_1(s) - \mathbf{H}_2(s)\|_\infty = 3.01 \times 10^{-5}$.

Conclusions and Future Work

- $n >> 10^6$: Forces usage of Inexact Solves in Krylov-based reduction
- Perturbation effects:
  - Backward and forward error analysis framework
  - *Good/Optimal* shift selection robust with respect to inexact solves
  - **I-IRKA**
    - (Locally) optimal reduced models for $n > 10^6$ without user intervention
    - Acceleration strategies
- Open issues:
  - Global $\mathcal{H}_2$ and/or $\mathcal{H}_\infty$ perturbation effects
  - Modifications to GMRES, effective preconditioning strategies