Canonical Dual Control for Nonconvex Distributed-Parameter Systems: Theory and Method[1]

David Yang Gao
Department of Mathematics
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061, USA. E-mail: gao@math.vt.edu

Abstract

This paper presents a potentially powerful canonical dual transformation method and associated duality theory for solving fully nonlinear distributed-parameter control problems. The extended Lagrange duality and the interesting triality theory proposed recently in finite deformation theory are generalized into nonconvex dynamical systems. A bifurcation criterion is proposed, which leads to an effective dual feedback control against the chaotic vibration in Duffing system.

1 Problems and Motivations

We shall study a duality approach for solving the following very general abstract distributed parameter problem (($\mathcal{P}$) for short),

$$(\mathcal{P}): \quad \rho u_{,tt} + A(u,\mu) = 0 \ \ \forall u \in \mathcal{U}_k, \tag{1}$$

where the feasible space $\mathcal{U}_k$ is a convex, non-empty subset of a reflexive Banach space $\mathcal{U}$ over an open space-time domain $\Omega_t = \Omega \times (0, t_c) \subset \mathbb{R}^n \times \mathbb{R}^+$, in which, certain essential boundary-initial conditions are prescribed. We assume that for a given distributed parameter control field $\mu(x,t)$ over Ω_t, the mapping $A(u,\mu)$ is a potential operator from $\mathcal{U}_k$ into its dual space $\mathcal{U}^*$, i.e., there exists a Gâteaux differentiable potential functional $P_\mu(u) = P(u;\mu)$, such that the directional derivative of P at $\bar{u} \in \mathcal{U}_k$ in the direction δu can be written as

$$\delta P_\mu(\bar{u}; \delta u) = \langle DP_\mu(\bar{u}), \delta u \rangle \ \ \forall \delta u \in \mathcal{U}_k,$$

where the operator $DP_\mu(\bar{u}) = A(\bar{u},\mu)$ is the Gâteaux derivative of P_μ at the point $\bar{u}$; the bilinear form $\langle \cdot, \cdot \rangle : \mathcal{U} \times \mathcal{U}^* \to \mathbb{R}$ places $\mathcal{U}$ and $\mathcal{U}^*$ in duality. By nonlinear operator theory we know that the mapping $A : \mathcal{U}_k \to \mathcal{U}^*$ is monotone if and only if P is convex on $\mathcal{U}_k$.

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The problem $(\mathcal{P})$ is said to be *exactly controllable* if for certain given initial data $(u_0(x), v_0(x))$ in $\mathcal{U}_k$ and the final state $(\bar{u}_c(x), \bar{v}_c(x))$ there exists suitable control function $\mu(x,t)$ such that the solution $u(x,t)$ of the problem $(\mathcal{P})$ satisfies

$$u(x, t_c) = \bar{u}_c(x), \;\; u_{,t}(x, t_c) = \bar{v}_c(x) \;\; \forall x \in \Omega. \tag{2}$$

Dually, the problem $(\mathcal{P})$ is said to be *observable* if for certain given input control $\mu(x,t)$, there exists an output function $h(u)$ such that the initial state $(u_o(x), v_o(x))$ can be uniquely determined from the output $z = h(u(x,t))$ over any interval $0 < t < t_c$.

The abstract form of problem $(\mathcal{P})$ covers a great variety of situations. Very often, the total potential $P_\mu(u)$ can be written as

$$P_\mu(u) = \Phi_\mu(u, \Lambda(u)) = W_\mu(\Lambda(u)) - F_\mu(u),$$

where Λ is a Gâteaux differentiable operator from $\mathcal{U}$ into another Banach space $\mathcal{E}$; the functional $W_\mu(\xi)$ is the so-called stored (or internal) potential; while the functional $F_\mu(u)$ represents the external potential of the system.

In convex Hamilton systems, the total potential $P_\mu(u)$ is convex and its Gâteaux derivative $A(u;\mu) = DP_\mu(u)$ is usually an elliptic operator in conservative problems. In linear field theory of mathematical physics, Λ is usually a gradient-like operator, say $\Lambda = \text{grad}$, and $W_\mu(\xi)$ is a quadratic functional, for example,

$$P_\mu(u) = \int_\Omega \frac{1}{2} a(x) |\nabla u|^2 \, \mathrm{d}\Omega - F_\mu(u),$$

where $a(x) > 0 \;\; \forall x \in \Omega$. In this case, the governing equation (1) reads

$$\rho u_{,tt} = \nabla \cdot (a(x) \nabla u) + DF_\mu(u) \;\; \forall (x,t) \in \Omega_t. \tag{3}$$

It is a linear wave equation if $F_\mu(u)$ is a linear functional, say $F_\mu(\mu) = \langle u \;,\; u^*(\mu) \rangle$, where $u^*(\mu)$ is a given function of the input control field $\mu(x,t)$. If $F_\mu(u)$ is nonlinear, then the governing equation (3) is semi-linear. In boundary control problems, the distributed-parameter μ also appears in the feasible set $\mathcal{U}_k$. In applications of engineering mechanics, the state variable u could be also a vector-valued function and Λ is a tensor type operator. For example, in the shear control of extended beam structures, the actuators are filaments attached to the upper and lower beam surfaces ($y = \pm h$). The external signals effect a change of the properties of these filaments in such way that they produce shear forces $\mu^{\pm}(x,t)$. Thus, $\mu^{\pm}(x,t)$ is, in effect, the applied distributed-control, and the composite beam/actuator system is then an instance of an active, or "smart" structure. Since the repeated operation of these actuator devices results large shear deformations, the traditional Timoshenko beam model can not be used to the study of these phenomena because it assumes that the shear deformation is a function of x and t alone and does not vary in the lateral beam direction. In order to study the control problems of smart structures, several extended beams models have been proposed recently by Gao and Russell (1994, 1996), where the state variable space $\mathcal{U} = C^1(\Omega_t; \mathbb{R}^2)$ is a displacement space over the space time domain $\Omega_t = (0, \ell) \times (-h, h) \times (0, t_c)$. The element $u = \{\chi(x, y, t), w(x, t)\} \in \mathcal{U}$ is a continuous, differentiable vector in $\mathbb{R}^2$ with domain Ω_t, where $\chi(x, y, t)$ measures the shear deformation of the beam at the point (x, y), while $w(x, t)$ is the deflection of the beam. In the case that the elastic beam subjected to the transverse load $f(x, t)$ undergone infinitesmall deformation, the total potential is a quadratic functional

$$\begin{aligned} P_\mu(\chi, w) \;\; &= \;\; \frac{1}{2} \int_\Omega [\chi_{,x}^2 + \beta(\chi_{,y} + w_{,x})^2] \, \mathrm{d}\Omega \\ & \quad - \int_0^\ell (\mu^+(x, t)\chi(x, h, t) + \mu^-(x, t)\chi(x, -h, t) + f(x, t)w) \, \mathrm{d}x. \end{aligned}$$

If the beam is clamped at $x = 0$ and simply supported at $x = \ell$, and subjected to a compressive load at $x = \ell$, the kinematical admissible space $\mathcal{U}_k \subset \mathcal{U}$ can be definded as

$$\mathcal{U}_k = \left\{ \left(\begin{array}{c} \chi \\ w \end{array} \right) \in \mathcal{U} \;\middle|\; \begin{array}{l} \chi(x, -y, t) = -\chi(x, y, t), \;\; w(0, t) = w(\ell, t) = 0, \\ \chi(0, y, t) = \chi_{,x}(\ell, y, t) = 0 \;\; \forall y \in [-h, h], \\ (\chi, w) = (\chi_0, w_0), \;\; (\chi_{,t}, w_{,t}) = (\dot{\chi}_0, \dot{w}_0) \;\; \forall (x, y) \in \Omega, \;\; t = 0 \end{array} \right\}.$$

In this case, the abstract governing equation (1) is a linear coupled partial differential system

$$\begin{aligned} &\rho_\chi \chi_{,tt} = \chi_{,xx} + \beta \chi_{,yy}, \\ &\rho_w w_{,tt} = \beta w_{,xx} + \tfrac{\beta}{2h}[\chi_{,x}(x, h, t) - \chi_{,x}(x, -h, t)] + f(x, t), \\ &\chi_{,y}(x, \pm h, t) + w_{,x}(x, t) = \pm\mu^\pm(x, t). \end{aligned} \tag{4}$$

Since the total potential of this system is strictly convex, for the given input control function $\mu^\pm(x, t)$, this system possesses a unique stable solution.

Due to the efforts of more than thirty years research by many well-known mathematicians and scientists, the mathematical theory for distributed-parameter control systems have been well-established for convex Hamilton systems governed by partial differential equations (cf. e.g., Russell, 1973, 1978, 1986, 1996;Chen *et al*, 1991; Komornik, 1994; Lasiecka and Triggiani, 1999) with substantial applications in mechanics and structures (see, for examples, Lagnese and Lions, 1988; Lasiecka, 1998a; Lasiecka and Triggiani, 1987, 1999; Zuazua, 1996). In linear systems, there exists a very elegant duality relationship between the controllability and observability (see Dolecki and Russell, 1977). If the system reversible, the well-known Russell principle states that the stabilizability implies its exact controllability. The celebrated review articles by Russell (1978) and Lions (1988) still serve the excellent introductions to the mathematical aspects of controllability, stabilization and perturbations for distributed-parameter systems.

Duality is a fundamental concept that underlies almost all natural phenomena. In classical optimization and calculus of variation, duality methods possess beautiful theoretical properties, potentially powerful alternative performances and pleasing relationships to many other fields. The associated theory and extremality principles have been well studied for convex static and Hamilton systems (cf. e.g., Toland, 1978, 1979; Auchmuty, 1983, 1989, 1997; Strang, 1986; Rockafellar and Wets, 1997). There is a growing interest in studying and applications of convex duality theory in optimal control (cf., e.g., Mossino (1975), Chan and Ho (1979), Chan (1985), Chan and Yung (1987), Barron (1990), Tanimoto (1992), Lee and Yung (1997), Bergounioux *et al* (1999), Arada and Raymond (1999) and many others). The interesting one-to-one analogy between the optimal control and engineering structural mechanics was discovered by Zhong *et al* (1993, 1999). Recently, the so-called primal-dual interior-point (PDIP) method has been considered as a revolution in linear constrained optimization problems (cf. e.g., Gay *et al*, 1998; Wright, 1998). It was shown by Helton *et al* (1998) that the fundamental H^∞ optimization problem of control can be naturally treated with the PDIP methods.

However, the beautiful duality relationship in convex systems is broken in nonconvex problems. In many applications of engineering and sciences, the total potential of system is usually nonconvex, and even nonsmooth. The exact controllability and stability for nonconvex/nonsmooth systems are much more difficult. For example, in the well-known von Kármán thin plate model, the state variable u is a vector-valued function $u = \{\chi(x,t), w(x,t)\}$ over $\Omega_t \subset \mathbb{R}^2 \times \mathbb{R}$, where $\chi = \{\chi_\alpha\}$ $(\alpha = 1,2)$ is an in-plane displacement vector, while $w(x,t)$ stands for the deflection of the plate at $(x,t) \in \Omega_t$. The total potential is a nonlinear functional

$$P(\chi, w) = \frac{1}{2} a(w,w) + \frac{1}{2} b(\xi(\chi,w), \xi(\chi,w)) - \int_\Omega f w \, \mathrm{d}\Omega, \tag{5}$$

where $a(w,w)$ and $b(\xi,\xi)$ are two bilinear forms, defined respectively by

$$\begin{aligned} a(w,w) &= K \int_\Omega [(1-\nu)(\nabla\nabla w)(\nabla\nabla w) + \nu \Delta w \Delta w] \, \mathrm{d}\Omega, \\ b(\xi,\xi) &= \int_\Omega h \xi_{\alpha\beta} C_{\alpha\beta\gamma\theta} \xi_{\gamma\theta} \, \mathrm{d}\Omega, \end{aligned}$$

and ξ is a Cauchy-Green type strain tensor, defined by

$$\xi_{\alpha\beta} = \frac{1}{2}(\chi_{\alpha,\beta} + \chi_{\beta,\alpha} + w_{,\alpha} w_{,\beta}), \quad \alpha, \beta = 1, 2.$$

The governing equations for dynamical von Kármán plate are coupled nonlinear partial differential system

$$\begin{aligned} &\rho_w w_{,tt} = h \nabla \cdot (\sigma \cdot \nabla w) - K_0 \Delta\Delta w + f, \\ &\rho_\chi \chi_{,tt} = \nabla \cdot \sigma, \quad \sigma = C \xi(\chi, w). \end{aligned} \tag{6}$$

This coupled nonlinear partial differential system is a typical example in finite deformation mechanics. The mathematical control theory for large deformation plates and shells has emerged as the most challenging and active research field in recent years. In a series of papers by Lasiecka and her colleagues (see, for examples, Horn and Lasiecka, 1994, 1995; Favini *et al*, 1996; Lasiecka, 1998, 1999), many important contributions and open questions have been addressed for stabilizability of the so-called *full von Kármán system* with nonlinear boundary feedback (see Lasiecka, 1998). A detailed documentation on mathematical control theory of coupled nonlinear PDE's has been given in a lecture note by Lasiecka (1999). Since the von Kármán model is valid only for plates subjected to the *moderately large* deflections, only the second-order nonlinear term $w_{,\alpha} w_{,\beta}$ is considered, and the governing equation is linearly dependent on the in-plane deformation χ. In many engineering applications, the acceleration term $\rho \chi_{,tt}$ can usually be ignored. Thus, the second equation in (6) reads $\nabla \cdot \sigma(\chi, w) = 0$. If the plate is subjected to compressive load on the boundary, the plate will be in the post-buckling state when the compressive load reaches its critical point. In this case, the total potential is nonconvex (i.e. the so-called double-well energy) (see Gao, 1995). In one-dimensional problems, the in-plane equilibrium condition $\sigma_{,x} = 0$ leads to a constant stress $\sigma = -\lambda$ everywhere in the domain $\Omega = (0, \ell) \subset \mathbb{R}$. In this case, the nonlinear von Kármán model (6) in $\mathbb{R}^2$ reduces to a linear equation in one-dimensional "beam" problem, i.e.,

$$\rho_w w_{,tt} = h\lambda w_{,xx} - K_0 w_{,xxxx} + f.$$

The main reason behind this von Kármán "paradox" is that the seconder order nonlinear term $w_{,\alpha} w_{,\beta}$ is considered for in-plane strain ξ, but it is ignored in the thickness direction. It may be appropriate for thin plates, but for one-dimensional beam models, this is wrong! It is shown in Gao (1996) that the strain in the thickness direction of the beam is proportional to the second-order term $w_{,x}^2$, and cannot be ignored when the beam is subjected to moderately large rotations. Thus, an extended large deformation beam model was proposed as

$$\rho_w w_{,tt} + K_0 w_{,xxxx} - k_0(\lambda - \frac{1}{2} w_{,x}^2) w_{,xx} - f = 0 \quad \text{in } \Omega_t = (0, \ell) \times (0, t_c), \tag{7}$$

where $k_0 > 0$ is a positive material constant. The total potential energy associated with this nonlinear beam theory is a nonlinear functional

$$P(w) = \int_I \frac{1}{2} \left(K_0 w_{,xx}^2 + k_0 (\frac{1}{2} w_{,x}^2 - \lambda)^2 \right) \, \mathrm{d}x - \int_I f w \, \mathrm{d}x. \tag{8}$$

In static problem, if the beam is clamped at $x = 0$, simply supported at $x = \ell$, the kinematically admissible space $\mathcal{U}_k$ can be written as

$$\mathcal{U}_k = \{ w \in C^2(0, \ell) | \;\; w(0) = w_{,x}(0) = 0, \;\; w(\ell) = w_{,xx}(\ell) = 0 \}.$$

It is clear that for the given Euler pre-buckling load $\lambda_c > 0$,

$$\int_I K_0 w_{,xx}^2 \, \mathrm{d}x \geq \lambda_c \int_I w_{,x}^2 \, \mathrm{d}x, \quad \forall w \in \mathcal{U}_k.$$

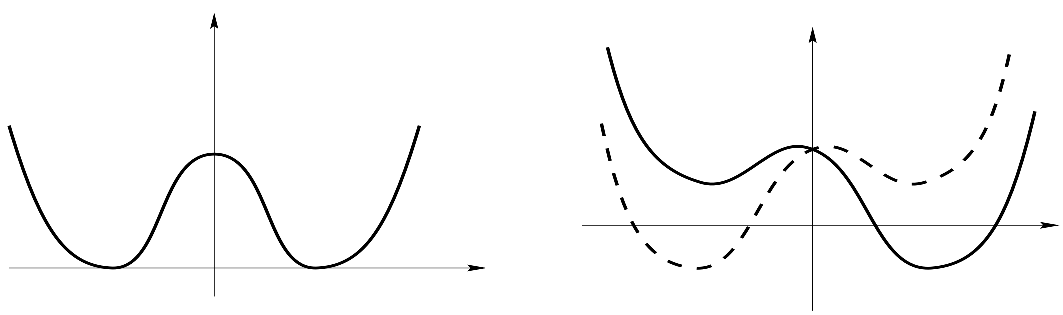
Thus, on $\mathcal{U}_k$,

$$\begin{aligned} P(w) &\geq \int_I \frac{1}{2}\left(\lambda_c w_{,x}^2 + k_0(\frac{1}{2}w_{,x}^2 - \lambda)^2\right) \, \mathrm{d}x - \int_I f w \, \mathrm{d}x \\ &= P_\mu(w) + \lambda \ell \lambda_c / k_0 - \ell \lambda_c^2/(2k_0^2), \end{aligned}$$

where $P_\mu(w)$ is a nonlinear functional

$$P_\mu(w) = \int_I \frac{1}{2} k_0 (\frac{1}{2} w_{,x}^2 - \mu)^2 \, \mathrm{d}x - \int_I f w \, \mathrm{d}x, \tag{9}$$

and $\mu = \lambda - \lambda_c/k_0 \in \mathbb{R}$ is a constant. Clearly, when the parameter $\mu > 0$, the stored energy $W_\mu(\epsilon) = \int_I \frac{1}{2} k_0 (\frac{1}{2}\epsilon^2 - \mu)^2 \, \mathrm{d}x$ is the well-known van der Waals double-well function (see Fig. 1a) of the linear strain $\epsilon = w_{,x}$, the beam is in a post-buckled (bifurcation) state. In this case, the total potential P_μ is nonconvex. It has three critical points: two local minimizers, corresponding to two possible stable buckled states, and one local maximizer, corresponding to an unstable buckled state. The global minimizer of P_μ depends on the lateral load f (see Fig. 1b).



(a) Graph of $W_\mu(\epsilon)$ (b) Graphs of $P_\mu(w)$ ($f > 0$ solid, $f < 0$ dashed)

Figure 1: Double-well energy and nonconvex potential

If the beam is subjected to a periodic dynamical load $f(x,t)$, the two local minimizers of P_μ become extremely unstable, and the beam is in dynamical post-buckling state. In this case, the governing equation (7) is replaced by

$$\rho_w w_{,tt} = k_0(\frac{3}{2} w_{,x}^2 - \mu) w_{,xx} + f(x,t) \quad \text{in } \Omega_t = (0,\ell) \times (0, t_c), \tag{10}$$

If the deflection $w(x,t)$ can be separated into $w(x,t) = u(t)v(x)$, this post-buckling dynamical beam model is equivalent to the well-known Duffing equation:

$$u_{,tt} = a u(\mu_o - \frac{1}{2}u^2) + \mu(t). \tag{11}$$

where $a > 0$ and $\mu_o \in \mathbb{R}$ are constants. This equation is extremely sensitive to the initial data. It is known that for certain give parameter μ_o and the driving input $\mu(t)$, this equation may produce the so-called *chaotic solutions*.

The problem of controlling chaotic systems is of significant practical importance and has attracted considerable attention during the last years. Mathematically speaking, the total potential of the chaotic system is usually nonconvex or even nonsmooth. Very small perturbations of the system's initial conditions and parameters may lead the system to different operating points with significantly different performance characteristics. This is the one of main reasons why the traditional perturbation analysis, the direct approaches and many standard control techniques cannot successfully be applied to chaotic systems. Based upon these observations and in order to handle the nonlinear problem, a school of new techniques has been developed (see, e.g., Fowler, 1989; Ott *et al*, 1990; Chen and Dong, 1992, 1993; Ogorzalek, 1993; Antoniou *et al*, 1996; Ghezzi and Piccardi, 1997; Mertzios and Koumboulis, 1996; Koumboulis and Mertzios, 2000). In the shear control of large deformation extended beam model, the equation (4) can be replaced by (see Gao, 2000a)

$$\begin{aligned}
&\chi_{,xx} + \beta\chi_{,yy} = 0, \\
&\rho_w w_{,tt} = \left(\tfrac{3\alpha^2}{2} w_{,x}^2 + \beta - \lambda\alpha\right) w_{,xx} + \tfrac{\beta}{2h}[\chi_{,x}(x,h,t) - \chi_{,x}(x,-h,t)] + f(x,t), \qquad (12) \\
&\chi_{,y}(x,\pm h,t) + w_{,x}(x,t) = \pm\mu^{\pm}(x,t),
\end{aligned}$$

where $\alpha > 0$ is a given material constant and $\lambda > 0$ represents the axial load. The total potential associated with this model is a nonconvex functional

$$\begin{aligned}
P_\mu(\chi, w) &= \frac{1}{2}\int_\Omega [(\chi_{,x}^2 + \frac{1}{2}\alpha w_{,x}^2 - \lambda)^2 + \beta(\chi_{,y} + w_{,x})^2]\, d\Omega \\
&\quad - \int_0^\ell (\mu^+(x,t)\chi(x,h,t) + \mu^-(x,t)\chi(x,-h,t) + f(x,t)w)\, dx.
\end{aligned}$$

In order to control the chaotic vibration of this nonconvex dynamical beam system, an efficient canonical dual feedback control method has been proposed recently by the author (Gao, 2000e).

The duality theory in fully nonlinear variational problems was originally studied by Gao and Strang (1989) for large deformation nonsmooth mechanics. In order to recover the broken symmetry in fully nonlinear systems (see Definition 2), a so-called *complementary gap function* was introduced. It was realized recently in post-buckling analysis of nonlinear beam theory (Gao, 1996) that this function recovered the duality gap between the nonconvex primal problems and the Fenchel-Rockafellar dual problems. A self-contained comprehensive presentation of the mathematical theory for general nonconvex systems was given recently by Gao (1999). A so-called *canonical dual transformation method* and associated triality theory have been proposed for solving nonconvex/nonsmooth variational-boundary value problems. Compared with the traditional analytic methods and direct approaches, the main advantages of this canonical dual transformation method are

(1) converting nonconvex/nonsmooth constrained variational problems into smooth unconstrained dual problems;

(2) transforming certain fully nonlinear partial differential equations into algebraic systems;

(3) providing powerful and efficient primal-dual alternative approaches.

The aim of this article is to generalize the author's previous results on nonconvex variational problems into distributed-parameter control systems. The rest of this paper is divided into four main sections. The next section set up the notation used in the paper. A general framework in fully nonlinear systems are discussed. Section 3 presents an extended Lagrangian critical point theorem and associated triality theory in general nonconvex dynamical systems. The critical points in fully nonlinear systems are classified. Section 4 is devoted mainly to the construction of dual action in fully nonlinear systems. The nice tri-duality proposed in static boundary value problems is generalized into control problems. Section 5 discusses the application in Duffing system. A bifurcation criterion is proposed which can be used for feedback controlling against chaotic vibrations.

2 Framework for Canonical Systems and Classification

Let $\mathcal{U}$ and $\mathcal{U}^*$ be two locally convex topological real linear spaces, placed in separating duality by a bilinear form $\langle \cdot, \cdot \rangle : \mathcal{U} \times \mathcal{U}^* \to \mathbb{R}$. Let $P : \mathcal{U}_s \to \mathbb{R}$ be a given functional, well-defined on a convex domain $\mathcal{U}_s \subset \mathcal{U}$ such that for any given $u \in \mathcal{U}_s$, $P(u)$ is Gâteaux differentiable. Thus, the Gâteaux derivative DP of P at $u \in \mathcal{U}_s$ is a mapping from $\mathcal{U}_s$ into $\mathcal{U}^*$. Let $\mathcal{U}_s^* \subset \mathcal{U}^*$ be the range of the mapping $DP : \mathcal{U}_s \to \mathcal{U}^*$. If the relation $u^* = DP(u)$ is reversible on $\mathcal{U}_s$, then, for any given $u^* \in \mathcal{U}_s^*$, the classical Legendre conjugate functional $P^* : \mathcal{U}_s^* \to \mathbb{R}$ of $P(u)$ is defined by

$$P^*(u^*) = \langle u(u^*), u^* \rangle - P(u(u^*)).$$

The conjugate pair (u, u^*) is called the *Legendre duality pair* on $\mathcal{U}_s \times \mathcal{U}_s^* \subset \mathcal{U} \times \mathcal{U}^*$ if and only if the equivelant relations

$$u^* = DP(u) \quad \Leftrightarrow \quad u = DP^*(u^*) \quad \Leftrightarrow \quad P(u) + P^*(u^*) = \langle u, u^* \rangle. \tag{13}$$

hold on $\mathcal{U}_s \times \mathcal{U}_s^*$.

The following notations and definitions, used in Gao (1999), will be of convenience in nonconvex control problems.

Definition 1 The set of functionals $P : \mathcal{U} \to \mathbb{R}$ which are either convex or concave is denoted by $\Gamma(\mathcal{U})$. In particular, let $\check{\Gamma}(\mathcal{U})$ denote the subset of functionals $P \in \Gamma(\mathcal{U})$ which are convex and $\hat{\Gamma}(\mathcal{U})$ the subset of $P \in \Gamma(\mathcal{U})$ which are concave.

The *canonical functional space* $\Gamma_G(\mathcal{U}_s)$ is a subset of functionals $P \in \Gamma(\mathcal{U}_s)$ which are Gâteaux differentiable on $\mathcal{U}_s \subset \mathcal{U}$, such that the relation $u^* = DP(u)$ is reversible for any given $u \in \mathcal{U}_s$. $\diamondsuit$

Clearly, if $P \in \Gamma_G(\mathcal{U}_s)$ and $\mathcal{U}_s^*$ is the range of the mapping $DP : \mathcal{U}_s \to \mathcal{U}^*$, then the Legendre duality relations (13) hold on $\mathcal{U}_s \times \mathcal{U}_s^*$.

Let $(\mathcal{E}, \mathcal{E}^*)$ be an another pair of locally convex topological real linear spaces paired in separating duality by the second bilinear form $\langle \cdot \, ; \, \cdot \rangle : \mathcal{E} \times \mathcal{E}^* \to \mathbb{R}$. The so-called *geometrical operator* $\Lambda : \mathcal{U} \to \mathcal{E}$ is a continuous, Gâteaux differentiable operator such that for any given $u \in \mathcal{U}_a \subset \mathcal{U}$, there exists an element $\xi \in \mathcal{E}_a \subset \mathcal{E}$ satisfying the *geometrical equation*

$$\xi = \Lambda(u).$$

The directional derivative of ξ at $\bar{u}$ in the direction $u \in \mathcal{U}$ is then definded by

$$\delta\xi(\bar{u}; u) := \lim_{\theta \to 0^+} \frac{\xi(\bar{u} + \theta u) - \xi(\bar{u})}{\theta} = \Lambda_t(\bar{u})u, \tag{14}$$

where $\Lambda_t(\bar{u}) = D\Lambda(\bar{u}) : \mathcal{U} \to \mathcal{E}$ denotes the Gâteaux derivative of the operator Λ at $\bar{u}$. For a given $\xi^* \in \mathcal{E}^*$, $G_\Lambda(u) = \langle \Lambda(u) \, ; \, \xi^* \rangle$ is a real-valued functional of u on $\mathcal{U}$. Its Gâteaux derivative at $\bar{u} \in \mathcal{U}$ in the direction $u \in \mathcal{U}$ reads

$$\delta G_\Lambda(\bar{u}; u) = \langle \Lambda_t(\bar{u})u \, ; \, \xi^* \rangle = \langle u \, , \, \Lambda_t^*(\bar{u})\xi^* \rangle,$$

where $\Lambda_t^*(\bar{u}) : \mathcal{E}^* \to \mathcal{U}^*$ is the adjoint operator of Λ_t associated with the two bilinear forms.

Let $\mathcal{V}$ and $\mathcal{V}^*$ be the velocity and momentum spaces, respectively, placed in duality by the third bilinear form $\langle * \, , \, * \rangle : \mathcal{V} \times \mathcal{V}^* \to \mathbb{R}$. For Newtonian systems, the kinetic energy $K : \mathcal{V} \to \mathbb{R}$ and its Legendre conjugate $K^* : \mathcal{V}^* \to \mathbb{R}$ are quadratic forms

$$K(v) = \int_\Omega \frac{1}{2}\rho v^2 \, d\Omega, \quad K^*(p) = \int_\Omega \frac{1}{2}\rho^{-1} p^2 \, d\Omega.$$

Thus the canonical physical relations between $\mathcal{V}$ and $\mathcal{V}^*$ are linear:

$$p = DK(v) = \rho v \quad \Leftrightarrow \quad v = DK^*(p) = \rho^{-1} p.$$

Let $\mathcal{V}_a \subset \mathcal{V}$ be a subspace defined by

$$\mathcal{V}_a = \{ v \in \mathcal{V} | \;\; v(x, 0) = v_0 \;\; \forall x \in \Omega \}. \tag{15}$$

Finally, we let $\mathcal{M}$ be an admissible control space over Ω_t. For any given $\mu \in \mathcal{M}$, we assume that there exists a Gâteaux differentiable functional $\Phi_\mu : \mathcal{U}_a \times \mathcal{E}_a \subset \mathcal{U} \times \mathcal{E} \to \mathbb{R}$, such that the total potential $P(u;\mu)$ of the system can be written as

$$P_\mu(u) = P(u;\mu) = \Phi_\mu(u, \Lambda(u)), \tag{16}$$

and the total action of the system

$$\Pi_\mu(u) = \int_0^{t_c} [K(u_{,t}) - \Phi_\mu(u, \Lambda(u))] \, \mathrm{d}t \tag{17}$$

is well-defined on the feasible space $\mathcal{U}_k$ given by

$$\mathcal{U}_k = \{u \in \mathcal{U}_a | \ \Lambda(u) \in \mathcal{E}_a, \ \ u_{,t} \in \mathcal{V}_a\}. \tag{18}$$

The following classification for distributed parameter control systems was originally introduced in nonlinear variational/boundary value problems by Gao (1998, 1999).

Definition 2 Suppose that for the problem $(\mathcal{P})$ given in (1), the associated total potential $P_\mu(u)$ is well-defined on its domain $\mathcal{U}_s \subset \mathcal{U}$. If the geometrical operator $\Lambda : \mathcal{U} \to \mathcal{E}$ can be chosen in such a way that $P_\mu(u) = \Phi_\mu(u, \Lambda(u))$, $\Phi_\mu \in \Gamma_G(\mathcal{U}_a) \times \Gamma_G(\mathcal{E}_a)$ and $\mathcal{U}_s = \{u \in \mathcal{U}_a | \ \Lambda(u) \in \mathcal{E}_a\}$. Then

(1) the transformation $\{P;\mathcal{U}_s\} \to \{\Phi_\mu;\mathcal{U}_a \times \mathcal{E}_a\}$ is called the *canonical transformation*, and $\Phi_\mu : \mathcal{U}_a \times \mathcal{E}_a \to \mathbb{R}$ is called the *canonical functional associated with* Λ;

(2) the problem $(\mathcal{P})$ is called *geometrically nonlinear (or linear)* if $\Lambda : \mathcal{U} \to \mathcal{E}$ is nonlinear (or linear); it is called *physically nonlinear* (resp. linear) if the duality mapping $D\Phi_\mu : \mathcal{U}_a \times \mathcal{E}_a \to \mathcal{U}_a^* \times \mathcal{E}_a^*$ is nonlinear (resp. linear); it is called *fully nonlinear* if it is both geometrically and physically nonlinear. ♢

The canonical transformation plays a fundamental role in duality theory of nonconvex systems. Clearly, if $\Phi_\mu \in \Gamma_G(\mathcal{U}_a) \times \Gamma_G(\mathcal{E}_a)$ is a canonical functional, the Gâteaux derivative $D\Phi_\mu : \mathcal{U}_a \times \mathcal{E}_a \to \mathcal{U}_a^* \times \mathcal{E}_a^* \subset \mathcal{U}^* \times \mathcal{E}^*$ is a monotone mapping, i.e., the duality relations

$$u^* = D_u\Phi_\mu(u, \xi), \quad \xi^* = D_\xi\Phi_\mu(u, \xi) \tag{19}$$

are reversible between the paired spaces $(\mathcal{U}_a, \mathcal{U}_a^*)$ and $(\mathcal{E}_a, \mathcal{E}_a^*)$, where $D_u\Phi_\mu$ and $D_\xi\Phi_\mu$ denote the partial Gâteaux derivatives of Φ_μ with respect to u and ξ, respectively. Thus, on $\mathcal{U}_k$, the directional derivative of P_μ at $\bar{u}$ in the direction $u \in \mathcal{U}_k$ can be written as

$$\begin{aligned} \delta P_\mu(\bar{u}; u) &= \langle u \ , \ D_u\Phi_\mu(\bar{u}, \Lambda(\bar{u}))\rangle + \langle \Lambda_t(\bar{u})u \ ; \ D_\xi\Phi_\mu(\bar{u}, \Lambda(\bar{u}))\rangle \\ &= \langle u \ , \ \bar{u}^*\rangle + \langle u \ ; \ \Lambda_t^*(\bar{u})\bar{\xi}^*\rangle \quad \forall u \in \mathcal{U}_k. \end{aligned}$$

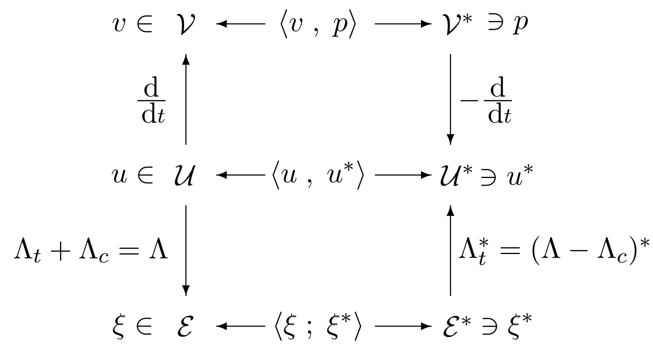


Figure 2: Framework in fully nonlinear Newtonian systems

In terms of canonical variables, the governing equation (1) for the fully nonlinear problems can be written in the *tri-canonical forms*, namely,

$$\begin{array}{lll} \text{(1) geometrical equations:} & v = u_{,t}, \;\; \xi = \Lambda(u), & \\ \text{(2) physical relations:} & p = \rho v, \;\; (u^*, \xi^*) = D\Phi_\mu(u, \xi), & \quad (20) \\ \text{(3) balance equation:} & p_{,t} + u^* + \Lambda_t^*(u)\xi^* = 0. & \end{array}$$

The framework for the fully nonlinear system is shown in Fig. 1. Extensive illustrations of the canonical transformation and the tri-canonical forms in mathematical physics and variational analysis were given in the monograph by Gao (1999).

In geometrically linear systems, where $\Lambda : \mathcal{U} \to \mathcal{E}$ is linear, we have $\Lambda = \Lambda_t$. For dynamical problems, if the total potential P_μ is convex, the total action associated with the problem $(\mathcal{P})$ is a *d.c. functional*, i.e., the difference of convex functionals:

$$\Pi_\mu(u) = \int_0^{t_c} [K(u_{,t}) - P_\mu(u)] \, \mathrm{d}t.$$

It was shown by Gao (1999) that the critical point of Π_μ either minimizes or maximizes Π_μ over the kinetically admissible space. The classical Hamiltonian associated with this d.c. functional Π_μ is a convex functional on the phase space $\mathcal{U} \times \mathcal{V}^*$, i.e.

$$H(u, p) = K^*(p) + P_\mu(u), \tag{21}$$

The classical canonical forms for convex Hamilton systems are well-known

$$\frac{\mathrm{d}}{\mathrm{d}t} u = D_p H(u, p), \quad -\frac{\mathrm{d}}{\mathrm{d}t} p = D_u H(u, p).$$

Furthermore, if $\Phi_\mu(u, \xi) = \frac{1}{2}\langle \xi \, ; \, C\xi \rangle - \langle u \, , \, \mu \rangle$ is a quadratic quadratic functional, where $C : \mathcal{E} \to \mathcal{E}^*$ is a linear operator, then the governing equations for linear system can be written as

$$\rho\bar{u}_{,tt} + \Lambda^* C \Lambda u = \mu.$$

For conservative systems, the operator $\Lambda^* C \Lambda$ is usually symmetric.

In geometrically nonlinear systems, $\Lambda \neq \Lambda_t$, and the total potential $P_\mu(u)$ is usually a nonconvex functional. In this case, we have the following operator decomposition

$$\Lambda(u) = \Lambda_t(u)u + \Lambda_c(u), \tag{22}$$

where $\Lambda_c : \mathcal{U} \to \mathcal{E}$ is called the complementary operator of the Gâteaux derivative operator Λ_t. By this decomposition, we have

$$\langle \Lambda(\bar{u}) \; ; \; \bar{\xi}^* \rangle = \langle \bar{u} \; , \; \Lambda_t^*(\bar{u})\bar{\xi}^* \rangle - G(\bar{u}, \bar{\xi}^*) \tag{23}$$

where $G : \mathcal{U} \times \mathcal{E}^* \to \mathbb{R}$ is so-called *complementary gap functional*, defined by

$$G(u, \xi^*) = \langle -\Lambda_c(u) \; ; \; \xi^* \rangle : \mathcal{U} \times \mathcal{E}^* \to \mathbb{R}. \tag{24}$$

This functional was first introduced by Gao and Strang (1989) in finite deformation theory, which plays a key role in nonconvex variational problems.

As a typical example in nonconvex dynamical systems, let us consider the post-buckling dynamical beam model (10) discussed in section 1. For a given feasible space $\mathcal{U}_k$, we consider the following nonconvex variational problem over the domain $\Omega_t = (0, \ell) \times (0, t_c)$

$$\Pi_\mu(u) = \int_{\Omega_t} \left[\frac{1}{2}\rho u_{,t}^2 - \frac{1}{2} a (\frac{1}{2} u_{,x}^2 - \mu)^2 + uf \right] \, dx \, dt \quad \to \text{sta} \;\; \forall u \in \mathcal{U}_k, \tag{25}$$

where a, μ are given positive constants. This nonconvex problem also appears very often in phase transitions and hysteresis.

First, we let $\Lambda = d/\, dx$ be a linear operator, and $P_\mu(u) = W_\mu(\Lambda u) - F_\mu(u)$ with

$$W_\mu(\epsilon) = \int_0^\ell \frac{1}{2} a (\frac{1}{2}\epsilon^2 - \mu)^2 \, dx, \quad F(u) = \int_0^\ell uf \, dx,$$

Thus, $W_\mu(\epsilon)$ is the so-called van der Waals' double-well function of the linear "strain" $\epsilon = u_{,x}$. Since $W_\mu(\epsilon)$ is not a canonical functional, the constitutive equation $\epsilon^* = DW_\mu(\epsilon)$ is not one-to-one. Thus, the Legendre conjugate of $W_\mu(\epsilon)$ does not have a simple algebraic expression. The Fenchel conjugate $W_\mu^*(\epsilon^*)$ of the double-well energy $W_\mu(\epsilon)$, defined by

$$W_\mu^*(\epsilon^*) = \sup_\epsilon \{ \langle \epsilon \; ; \; \epsilon^* \rangle - W_\mu(\epsilon) \},$$

is always a convex, lower semi-continuous functional. However, the well-known Fenchel-Young inequality

$$W_\mu(u_{,x}) \geq \langle u_{,x} \; ; \; \epsilon^* \rangle - W_\mu^*(\epsilon^*)$$

leads to a so-called duality gap between the primal problem and the Fenchel-Rockafellar dual problem (see Gao, 1999). This nonzero duality gap indicates that the well-established Fenchel-Rockafellar duality theory can only be used for solving convex variational problems.

From the theory of continuum mechanics we know that in finite deformation problems, $\epsilon = u_{,x}$ is not a strain measure (it does not satisfy the *axiom of material frame-indifference* (cf. e.g., Gao, 1999). In order to recover this duality gap, we need to choose a suitable geometrical operator Λ, say, $\Lambda(u) = \frac{1}{2}u_{,x}^2 - \mu$, so that the nonconvex problem (25) can be put in our framework. In continuum mechanics, this quadratic measure $\xi = \Lambda(u)$ is a Cauchy-Green type strain. Thus, in terms of u and ξ, $\Phi_\mu(u, \xi) = W_\mu(\xi) - F_\mu(u) = \frac{1}{2}\langle \xi \ ; \ a\xi \rangle - \langle u \ , \ f \rangle$ is a canonical functional. The Legendre conjugate of the quadratic functional $W_\mu(\xi) = \frac{1}{2}\langle \xi \ ; \ a\xi \rangle$ is simply defined by $W^*(\xi^*) = \frac{1}{2}\langle a^{-1}\xi^* \ ; \ \xi^* \rangle$. The operator decomposition (22) for this quadratic operator reads

$$\Lambda(u) = \Lambda_t(u)u + \Lambda_c(u), \ \ \Lambda_t(u)u = u_{,x}u_{,x}, \ \ \Lambda_c(u) = -\frac{1}{2}u_{,x}^2 - \mu.$$

The complementary gap functional associated with this quadratic operator is a quadratic functional of u

$$G(u, \xi^*) = \langle -\Lambda_c(u) \ ; \ \xi^* \rangle = \int_0^\ell \frac{1}{2}u_{,x}^2 \xi^* \ \mathrm{d}x.$$

For homogeneous boundary conditions, we have

$$\langle \Lambda_t(u)u \ ; \ \xi^* \rangle = \int_0^\ell u_{,x}u_{,x}\xi^* \ \mathrm{d}x = -\int_0^\ell u(u_{,x}\xi^*)_{,x} \ \mathrm{d}x = \langle u \ , \ \Lambda_t^*(u)\xi^* \rangle,$$

which leads to the adjoint operator Λ_t^* of Λ_t. Thus, the tri-canonical equations for this nonconvex problem can be listed as the following.

$$v = u_{,t}, \ \ \xi = \frac{1}{2}au_{,x}^2 - \mu,$$

$$p = \rho v, \ \ \xi^* = DW_\mu(\xi) = a\xi, \ \ u^* = DF_\mu(u) = f$$

$$p_{,t} = -\Lambda_t^*(u)\xi^* + u^* = (u_{,x}\xi^*)_{,x} + f.$$

Since the geometrical operator Λ is nonlinear, and the canonical constitutive equations are linear, the nonconvex problem (25) is a geometrically nonlinear system.

3 Extended Lagrangian and Triality Theory

The triality theory in nonconvex problems was originally proposed by the author (Gao, 1996, 1997, 1999, 2000) in static finite deformation theory and global optimization. In this section, we will generalize this interesting result into fully nonlinear dynamical systems.

We assume that for a given fully nonlinear system, there exists a Gâteaux differentiable operator $\Lambda : \mathcal{U}_a \to \mathcal{E}_a$ such that the total potential of the system can be written as

$$P_\mu(u) = W_\mu(\Lambda(u)) - F_\mu(u), \tag{26}$$

where $W_\mu \in \check{\Gamma}_G(\mathcal{E}_a)$ is a convex canonical functional, while $F_\mu : \mathcal{U}_a \to \mathbb{R}$ is a linear functional. Thus, the primal problem $(\mathcal{P})$ can be reformulated as the following.

Problem 1 (Primal Distributed-Parameter Control Problem) For any given primal feasible space $\mathcal{U}_k = \{u \in \mathcal{U}_a | \ \ u_{,t} \in \mathcal{V}_a, \ \ \Lambda(u) \in \mathcal{E}_a\}$ and the final state $(\bar{u}_c(x), \bar{v}_c(x))$, find the control field $\mu(x,t) \in \mathcal{M}$ such that the solution $\bar{u}(x,t)$ of the variational problem

$$(\mathcal{P}) : \quad \Pi_\mu(u) = \int_0^{t_c} [K(u_{,t}) - W_\mu(\Lambda(u)) + F_\mu(u)] \, \mathrm{d}t \to \text{sta} \ \ \forall u \in \mathcal{U}_k \tag{27}$$

satisfying the controllability condition

$$(\bar{u}(x,t_c), \bar{u}_{,t}(x,t_c)) = (\bar{u}_c(x), \bar{v}_c(x)) \ \ \forall x \in \Omega.$$

It is easy to check that the critical point condition $D\Pi_\mu(\bar{u}) = 0$ leads to the the canonical governing equation

$$\rho \bar{u}_{,tt} = DF_\mu(\bar{u}) - \Lambda_t^*(\bar{u}) DW_\mu(\Lambda(\bar{u})). \tag{28}$$

By the Legendre-Fenchel transformation, the conjugate of $W_\mu(\xi)$ is defined by

$$W_\mu^*(\xi^*) = \sup_{\xi \in \mathcal{E}} \{\langle \xi \, ; \, \xi^* \rangle - W_\mu(\xi)\}.$$

Since $W_\mu : \mathcal{E}_a \to \mathbb{R}$ is a convex canonical functional, $W_\mu^*(\xi^*)$ is well-defined on the range $\mathcal{E}_a^*$ of the duality mapping $DW_\mu^* : \mathcal{E}_a \to \mathcal{E}^*$, the Legendre duality relation

$$\xi^* = DW_\mu(\xi) \Leftrightarrow \xi = DW_\mu^*(\xi^*) \Leftrightarrow W_\mu(\xi) + W_\mu^*(\xi) = \langle \xi \, ; \, \xi^* \rangle$$

holds on $\mathcal{E}_a \times \mathcal{E}_a^*$. Moreover, we have $W_\mu^{**}(\xi) = W_\mu(\xi)$ for all $\xi \in \mathcal{E}_a$. Let $\mathcal{Z} = \mathcal{U} \times \mathcal{V}^* \times \mathcal{E}^*$ be the so-called *extended canonical phase space.*

Definition 3 Suppose that for a given problem $(\mathcal{P})$, there exists a Gâteaux differentiable operator $\Lambda : \mathcal{U} \to \mathcal{E}$ and canonical functionals $W_\mu \in \Gamma(\mathcal{E})$, $F_\mu \in \Gamma(\mathcal{U})$ such that $P_\mu(u) = W_\mu(\Lambda(u)) - F_\mu(u)$. Then

(1) the functional $H_\mu : \mathcal{Z} \to \mathbb{R}$ defined by

$$H_\mu(u, p, \xi^*) = K^*(p) - W_\mu^*(\xi^*) + F_\mu(u) \in \Gamma(\mathcal{U}) \times \Gamma(\mathcal{V}^*) \times \Gamma(\mathcal{E}^*) \tag{29}$$

is called the *extended canonical Hamiltonian density* associated with Π_μ;

(2) the functional $L_\mu : \mathcal{Z} \to \mathbb{R}$ definded by

$$L_\mu(u, p, \xi^*) = \langle u_{,t}\ ,\ p\rangle - \langle \Lambda(u)\ ;\ \xi^*\rangle - H_\mu(u, p, \xi^*) \tag{30}$$

is called the *extended Lagrangian density* of $(\mathcal{P})$ associated with Λ;

(3) the functional $\Xi_\mu : \mathcal{Z} \to \mathbb{R}$ definded by

$$\Xi_\mu(u, p, \xi^*) = \int_0^{t_c} L_\mu(u, p, \xi^*)\, \mathrm{d}t \tag{31}$$

is called the *extended Lagrangian form* of $(\mathcal{P})$. It is called the *canonical Lagrangian form* if $\Xi_\mu \in \Gamma(\mathcal{U}) \times \Gamma(\mathcal{V}^*) \times \Gamma(\mathcal{E}^*)$. $\diamondsuit$

A point $(\bar{u}, \bar{p}, \bar{\xi}^*) \in \mathcal{Z}$ is said to be a critical point of Ξ_μ if Ξ_μ is Gâteaux-differentiable at $(\bar{u}, \bar{p}, \bar{\xi}^*)$ and $D\Xi_\mu(\bar{u}, \bar{p}, \bar{\xi}^*) = 0$. It is easy to find out that the criticality condition $D\Xi_\mu(\bar{u}, \bar{p}, \bar{\xi}^*) = 0$ leads to the following *canonical Lagrange equations*

$$D\Xi_\mu(\bar{u}, \bar{p}, \bar{\xi}^*) = 0 \quad \Rightarrow \left\{ \begin{array}{l} \Lambda(\bar{u}) = D_{\xi^*} W_\mu^*(\bar{\xi}^*), \quad \bar{u}_{,t} = DK^*(\bar{p}), \\ \bar{p}_{,t} = DF_\mu(\bar{u}) - \Lambda_t^*(\bar{u})\bar{\xi}^*. \end{array} \right. \tag{32}$$

Since W_μ and F_μ are canonical functionals, we know that, by the Legendre duality theory, any critical point of Ξ_μ solves the variational problem $(\mathcal{P})$.

Since $F_\mu(u) : \mathcal{U}_a \to \mathbb{R}$ is a linear functional, by the Riesz representation theory, there exists an element $\bar{u}^*(\mu) \in \mathcal{U}^*$ such that $F_\mu(u) = \langle u\ ,\ \bar{u}^*(\mu)\rangle$. Thus, the extended Lagrangian associated with $(\mathcal{P})$ can be written as

$$\Xi_\mu(u, p, \xi^*) = \int_0^{t_c} \left[\langle u_{,t}\ ,\ p\rangle - \langle \Lambda(u)\ ;\ \xi^*\rangle - K^*(p) + W^*(\xi^*) + \langle u\ ,\ \bar{u}^*(\mu)\rangle\right] \mathrm{d}t. \tag{33}$$

Note that $\Xi_\mu : \mathcal{V}_a^* \times \mathcal{E}_a^* \to \mathbb{R}$ is a saddle functional for any given $u \in \mathcal{U}_a$, we have always the equality

$$\inf_{\xi^* \in \mathcal{E}_a^*} \sup_{p \in \mathcal{V}_a^*} \Xi_\mu(u, p, \xi^*) = \sup_{p \in \mathcal{V}_a^*} \inf_{\xi^* \in \mathcal{E}_a^*} \Xi_\mu(u, p, \xi^*) \ \ \forall u \in \mathcal{U}_a. \tag{34}$$

However, for any given $(p, \xi^*) \in \mathcal{V}_a^* \times \mathcal{E}_a^*$, the convexity of $\Xi_\mu(\cdot, p, \xi^*) \to \mathbb{R}$ depends on the operator Λ. Let $\mathcal{L}_c \subset \mathcal{Z}_a = \mathcal{U}_a \times \mathcal{V}_a^* \times \mathcal{E}_a^*$ be a critical point set of Ξ_μ, i.e.

$$\mathcal{L}_c = \{(\bar{u}, \bar{p}, \bar{\xi}^*) \in \mathcal{Z}_a |\ \ \delta\Xi(\bar{u}, \bar{p}, \bar{\xi}^*; u, p, \xi^*) = 0 \ \ \forall (u, p, \xi^*) \in \mathcal{Z}_a\}.$$

For any given critical point $(\bar{u}, \bar{p}, \bar{\xi}^*) \in \mathcal{L}_c$, we let $\mathcal{Z}_r = \mathcal{U}_r \times \mathcal{V}_r^* \times \mathcal{E}_r^* \subset \mathcal{Z}_a$ be its *neighborhood* such that on $\mathcal{Z}_r$, $(\bar{u}, \bar{p}, \bar{\xi}^*)$ is the only critical point of Ξ_μ. The following extremum results are of fundamental importance in the stability analysis of nonlinear dynamical systems.

Theorem 1 (Triality Theorem) *Suppose that $(\bar{u}, \bar{p}, \bar{\xi}^*) \in \mathcal{L}_c$, and $\mathcal{Z}_r$ is a neighborhood of $(\bar{u}, \bar{p}, \bar{\xi}^*)$.*

If $\langle \Lambda(u) \; ; \; \bar{\xi}^* \rangle$ *is concave on* $\mathcal{U}_r$*, then on* $\mathcal{Z}_r$*,*

$$\Xi_\mu(\bar{u}, \bar{p}, \bar{\xi}^*) = \min_u \max_p \min_{\xi^*} \Xi_\mu(u, p, \xi^*) = \max_p \min_u \min_{\xi^*} \Xi_\mu(u, p, \xi^*). \tag{35}$$

However, if $\langle \Lambda(u) \; ; \; \bar{\xi}^* \rangle$ *is convex on* $\mathcal{U}_r$*, then on* $\mathcal{Z}_r$ *we have either*

$$\begin{aligned} \Xi_\mu(\bar{u}, \bar{p}, \bar{\xi}^*) &= \min_u \max_p \min_{\xi^*} \Xi_\mu(u, p, \xi^*) = \min_p \max_u \min_{\xi^*} \Xi_\mu(u, p, \xi^*) \\ &= \min_{\xi^*, u} \max_p \Xi_\mu(u, p, \xi^*) = \min_{p, \xi^*} \max_u \Xi_\mu(u, p, \xi^*). \end{aligned} \tag{36}$$

or

$$\begin{aligned} \Xi_\mu(\bar{u}, \bar{p}, \bar{\xi}^*) &= \max_u \min_{\xi^*} \max_p \Xi_\mu(u, p, \xi^*) = \max_p \min_{\xi^*} \max_u \Xi_\mu(u, p, \xi^*) \\ &= \min_{\xi^*} \max_{u,p} \Xi_\mu(u, p, \xi^*) = \max_{u,p} \min_{\xi^*} \Xi_\mu(u, p, \xi^*). \end{aligned} \tag{37}$$

Proof. Since $W^*_\mu \in \check{\Gamma}(\mathcal{E}^*_a)$, $K^* \in \check{\Gamma}(\mathcal{V}^*_a)$, if $\langle \Lambda(u) \; ; \; \bar{\xi}^* \rangle$ is concave on $\mathcal{U}_r$, then for the given $\bar{\xi}^*$, $\Xi_\mu \in \check{\Gamma}(\mathcal{U}_r) \times \hat{\Gamma}(\mathcal{V}^*_a)$ is a saddle functional. Thus the equality (35) follows from the saddle-Lagrangian duality theorem (cf. e.g., Gao, 1999). However, if $\langle \Lambda(u) \; ; \; \bar{\xi}^* \rangle$ is convex on $\mathcal{U}_r$, then for any given $\xi^* \in \mathcal{E}^*_r$, the extended Lagrangian $\Xi_\mu \in \hat{\Gamma}(\mathcal{U}_r) \times \hat{\Gamma}(\mathcal{V}^*_a)$ is a *super-critical functional* (see Gao, 1999). By the *super-Lagrangian duality theorem* proved in Gao (1999), we have either (36) or (37). □

4 Dual Action and Tri-Duality Theory

The goal of this section is to develop a dual approach for solving the distributed parameter control problem $(\mathcal{P})$. For any given $u \in \mathcal{U}_k$, the extended Lagrangian density $\Xi_\mu(u, p, \xi^*)$ is a saddle functional on $\mathcal{V}^* \times \mathcal{E}^*$, and we have

$$\Pi_\mu(u) = \sup_{p \in \mathcal{V}^*} \inf_{\xi^* \in \mathcal{E}^*} \Xi_\mu(u, p, \xi^*) \;\; \forall u \in \mathcal{U}_k. \tag{38}$$

On the other hand, the dual action $\Pi^d_\mu : \mathcal{V}^*_a \times \mathcal{E}^*_a \to \mathbb{R}$ can be defined by

$$\begin{aligned} \Pi^d_\mu(p, \xi^*) &= \text{sta}\{\Xi_\mu(u, p, \xi^*) | \;\; \forall u \in \mathcal{U}_a\} \\ &= F^\Lambda_\mu(p, \xi^*) - \int_0^{t_c} [K^*(p) - W^*_\mu(\xi^*)] \, dt, \;\; \forall (p, \xi^*) \in \mathcal{V}^*_a \times \mathcal{E}^*_a. \end{aligned} \tag{39}$$

where $F^\Lambda_\mu(p, \xi^*)$ is the so-called Λ-dual functional of $F_\mu(u)$ defined by

$$F^\Lambda_\mu(p, \xi^*) = \underset{u \in \mathcal{U}_a}{\text{sta}} \int_0^{t_c} [\langle u_{,t} \; , \; p \rangle - \langle \Lambda(u) \; ; \; \xi^* \rangle + F_\mu(u)] \, dt. \tag{40}$$

Since $F_\mu(u) = \langle u \ , \ \bar{u}^*(\mu)\rangle$ is a linear functional, for any given $(p, \xi^*) \in \mathcal{V}_a^* \times \mathcal{E}_a^*$ and the applied control $\mu \in \mathcal{M}$, the solution $\bar{u}$ of this stationary problem (40) satisfies the balance equation

$$p_{,t} + \Lambda_t^*(\bar{u})\xi^* = \bar{u}^*(\mu) \quad \text{in } \Omega_t. \tag{41}$$

For geometrically linear systems, where Λ is a linear operator, we have

$$F_\mu^\Lambda(p, \xi^*) = up|_{t=0}^{t=t_c}, \quad s.t. \ \ \Lambda^*\xi^* + p_{,t} = \bar{u}^*(\mu). \tag{42}$$

In this case,

$$\Pi_\mu^d(p, \xi^*) = up|_{t=0}^{t=t_c} + \int_0^{t_c} [W_\mu^*(\xi^*) - K^*(p)] \, \mathrm{d}t \tag{43}$$

is the classical complementary action in linear engineering dynamical systems (see Tabarrok and Rimrott, 1994) defined on the dual feasible space

$$\mathcal{T}_s = \{(p, \xi^*) \in \mathcal{V}_a \times \mathcal{E}_a^* | \ \ p_{,t} + \Lambda^*\xi^* = \bar{u}^*(\mu)\}.$$

In fully nonlinear systems, we let $\mathcal{T}_s \subset \mathcal{V}_a^* \times \mathcal{E}_a^*$ be a subspace such that for any given $(p, \xi^*) \in \mathcal{T}_s$, the critical point $\bar{u}$ can be determined by (41) as $\bar{u} = \bar{u}(p, \xi^*)$ and the dual action Π_μ^d is well defined by (39). Thus, by the operator decomposition $\Lambda = \Lambda_t + \Lambda_c$, we have

$$F_\mu^\Lambda(p, \xi^*) = up|_{t=0}^{t=t_c} + \int_0^{t_c} G^d(p, \xi^*) \, \mathrm{d}t, \ \ s.t. \ \ \Lambda_t^*(\bar{u})\xi^* + p_{,t} = u^*(\mu), \tag{44}$$

where $G^d(p, \xi^*) = \langle -\Lambda_c(\bar{u}) \ ; \ \xi^*\rangle$ is the so-called pure complementary gap functional. Then, the problem dual to the primal control problem $(\mathcal{P})$ can be proposed as the following.

Problem 2 (Dual Distributed-Parameter Control Problem) For a given dual feasible space $\mathcal{T}_s$ and the final state $(u_c(x), v_c(x))$, find the control field $\mu(x,t) \in \mathcal{M}$ such that the dual solution $(\bar{p}(x,t), \bar{\xi}^*(x,t))$ of the dual variational problem

$$(\mathcal{P}^d): \quad \Pi_\mu^d(p, \xi^*) \ \ \to \text{sta} \ \ \forall (p, \xi^*) \in \mathcal{T}_s \tag{45}$$

and the associated state $\bar{u}(x,t)$ satisfying the controllability condition

$$(\bar{u}(x, t_c), \rho^{-1}\bar{p}(x, t_c)) = (u_c(x), v_c(x)) \ \ \forall x \in \Omega. \tag{46}$$

The following lemma plays a key role in duality theory for nonlinear dynamical systems.

Lemma 1 *Let $\Xi_\mu(u, p, \xi^*)$ be a given extended Lagrangian associated with $(\mathcal{P})$ and $\Pi_\mu^d(p, \xi^*)$ the dual action defined by (39). Suppose that $\mathcal{Z}_r = \mathcal{U}_r \times \mathcal{V}_r^* \times \mathcal{E}_r^*$ is an open subset of $\mathcal{Z}_a$ and $(\bar{u}, \bar{p}, \bar{\xi}^*) \in \mathcal{Z}_r$ is a critical point of Ξ_μ on $\mathcal{Z}_r$, Π_μ is Gâteaux differentiable at $\bar{u}$, and Π_μ^d is Gâteaux differentiable at $(\bar{p}, \bar{\xi}^*)$. Then $D\Pi_\mu(\bar{u}) = 0$, $D\Pi_\mu^d(\bar{p}, \bar{\xi}^*) = 0$, and*

$$\Pi_\mu(\bar{u}) = \Xi_\mu(\bar{u}, \bar{p}, \bar{\xi}^*) = \Pi_\mu^d(\bar{p}, \bar{\xi}^*). \tag{47}$$

The proof of this lemma can be found in Gao (1998) in parametrical variational analysis. Lemma 4 shows that the critical points of the extended Lagrangian are also the critical points for both the primal and dual variational problems.

Theorem 2 (Tri-Duality Theorem) *Suppose that* $(\bar{u}, \bar{p}, \bar{\xi}^*) \in \mathcal{L}_c$ *is a critical point of* Ξ_μ *and* $\mathcal{Z}_r = \mathcal{U}_r \times \mathcal{V}_r^* \times \mathcal{E}_r^*$ *is a neighborhood of* $(\bar{u}, \bar{p}, \bar{\xi}^*)$ *such that* $\mathcal{V}_r^* \times \mathcal{E}_r^* \subset \mathcal{T}_s$*. If* $\langle \Lambda(u) \, ; \, \bar{\xi}^* \rangle$ *is concave on* $\mathcal{U}_r$*, then*

$$\Pi_\mu(\bar{u}) = \min_{u \in \mathcal{U}_r} \Pi_\mu(u) \quad \textit{iff} \quad \Pi_\mu^d(\bar{p}, \bar{\xi}^*) = \max_{p \in \mathcal{V}_r^*} \min_{\xi^* \in \mathcal{E}_r^*} \Pi_\mu^d(p, \xi^*). \tag{48}$$

However, if $\langle \Lambda(u) \, ; \, \bar{\xi}^* \rangle$ *is convex on* $\mathcal{U}_r$*, then*

$$\Pi_\mu(\bar{u}) = \min_{u \in \mathcal{U}_r} \Pi_\mu(u) \quad \textit{iff} \quad \Pi_\mu^d(\bar{p}, \bar{\xi}^*) = \min_{(p, \xi^*) \in \mathcal{T}_s} \Pi_\mu^d(p, \xi^*); \tag{49}$$

$$\Pi_\mu(\bar{u}) = \max_{u \in \mathcal{U}_r} \Pi_\mu(u) \quad \textit{iff} \quad \Pi_\mu^d(\bar{p}, \bar{\xi}^*) = \max_{p \in \mathcal{V}_r^*} \min_{\xi^* \in \mathcal{E}_r^*} \Pi_\mu^d(p, \xi^*). \tag{50}$$

Proof. This theorem can be proved by combining Lemma 1 and the triality theorem.□

5 Feedback Control Against Chaotic Duffing System

As a typical example, let us consider the very simple nonconvex dynamical problem over the time domain $I = (0, t_c)$

$$\Pi_\mu(u) = \int_I [\rho u'^2 - \frac{1}{2} a (\frac{1}{2} u^2 - \mu_o)^2 + \mu u] \, \mathrm{d}t \quad \to \mathrm{sta} \;\; \forall u \in \mathcal{U}_k. \tag{51}$$

The kinematically admissible space $\mathcal{U}_k$ for the initial-value problem of this one-dimensional dynamical system is given simply as

$$\mathcal{U}_k = \{ u \in \mathcal{L}^4(0, t_c) | \;\; u' \in \mathcal{L}^2(0, t_c), \;\; u(0) = u_0, \;\; u'(0) = v_0 \}.$$

The criticality condition for Π_μ leads to the well-known Duffing equation

$$\rho u'' = a u (\mu_o - \frac{1}{2} u^2) + \mu(t), \;\; \forall t \in I, \;\; u \in \mathcal{U}_k. \tag{52}$$

In terms of the nonlinear canonical measure $\xi = \Lambda(u) = \frac{1}{2} u^2$, the energy density $W_\mu(\xi)$ and its conjugate $W_\mu^*(\varsigma)$ are convex functions:

$$W_\mu(\xi) = \frac{1}{2} a (\xi - \mu_o)^2, \quad W_\mu^*(\varsigma) = \frac{1}{2a} \varsigma^2 + \mu_o \varsigma.$$

The extended Lagrangian for this nonconvex system is

$$\Xi_\mu(u, p, \varsigma) = \int_I \left(p u' - \varsigma (\frac{1}{2} u^2 - \mu_o) - \frac{1}{2\rho} p^2 + \frac{1}{2a} \varsigma^2 \right) \, \mathrm{d}t + \int_I \mu u \, \mathrm{d}t. \tag{53}$$

The criticality condition $D_u \Xi_\mu(\bar{u}, p, \varsigma) = 0$ leads to the equilibrium equation

$$p' + \bar{u}\varsigma = \mu \ \ \forall t \in I.$$

Clearly, the critical point $\bar{u} = (\mu - p')/\varsigma$ is well-defined for any nonzero ς. Thus, the dual feasible space can be defined as

$$\mathcal{T}_s = \left\{ (p, \varsigma) \in \mathcal{C}^1(I) \left| \begin{array}{l} p(0) = \rho v_0, \ \ -\mu_o a \le \varsigma(t) < +\infty, \\ \varsigma(t) \neq 0 \ \ \forall t \in I, \ \ \varsigma(0) = a(\frac{1}{2}u_0^2 - \mu_o) \end{array} \right. \right\}.$$

Substituting $\bar{u} = (\mu - p')/\varsigma$ into Ξ^d_μ, the dual action is obtained as

$$\begin{aligned} \Pi^d_\mu(p, \varsigma) &= \operatorname*{sta}_{u \in \mathcal{U}_a} \Xi_\mu(u, p, \varsigma) \\ &= p(t_c)u(t_c) - \rho v_0 u_0 + \int_I [\frac{1}{2a}\varsigma^2 + \mu_o \varsigma + \frac{(p' - \mu)^2}{2\varsigma} - \frac{1}{2\rho}p^2] \, \mathrm{d}t, \end{aligned} \tag{54}$$

which is well defined on $\mathcal{T}_s$. The criticality condition for Π^d_μ leads to the *dual Duffing system* in the time domain $I \subset \mathbb{R}$

$$\left(\frac{1}{\varsigma}(p' - \mu) \right)' + \frac{1}{\rho} p = 0, \tag{55}$$

$$\varsigma^2 (\frac{1}{a}\varsigma + \mu_o) = \frac{1}{2}(\mu - p')^2. \tag{56}$$

This system consists of the so-called *differential-algebraic equations* (DAE's), which arise naturally in many applications (cf., e.g., Brenan *et al*, 1996; Beardmore and Song, 1998). Although the numerical solution of these types of systems has been the subject of intense research activity in the past few years, the solvability of each problem depends mainly on the so-called *index* of the system. Clearly, the algebraic equation (56) has zero solution $\varsigma = 0$ if and only if $\sigma = (\mu - p') = 0$. Otherwise, for any nonzero $\sigma(t) = \mu(t) - p'(t)$, the algebraic equation (56) has at most three real roots $\varsigma_i(t)$ $(i = 1, 2, 3)$, each of them leads to the state solution $u_i(t) = (\mu(t) - p_i'(t))/\varsigma_i(t)$.

Theorem 3 (Stability and Bifurcation Criteria) *For a given parameter $\mu_o > 0$, initial data (u_0, v_0) and the input control $\mu(t)$, if at a certain time period $I_s \subset I = (0, t_c)$,*

$$\mu_c(t) = \frac{3}{2} \left(\frac{\mu(t) - p'(t)}{a} \right)^{2/3} > \mu_o, \ \ t \in I_s \tag{57}$$

then the Duffing system possesses only one solution set $(\bar{u}(t), \bar{p}(t), \bar{\varsigma}(t))$ satisfying $\bar{\varsigma}(t) > 0 \ \ \forall t \in I_s$, and over the period I_s,

$$\Pi_\mu(\bar{u}) = \min \Pi_\mu(u) \quad \textit{iff} \quad \Pi^d_\mu(\bar{p}, \bar{\varsigma}) = \min \Pi^d_\mu(p, \varsigma), \tag{58}$$

$$\Pi_\mu(\bar{u}) = \max \Pi_\mu(u) \quad \textit{iff} \quad \Pi^d_\mu(\bar{p}, \bar{\varsigma}) = \max_p \min_\varsigma \Pi^d_\mu(p, \varsigma). \tag{59}$$

However, if at a certain time period $I_b \subset I = (0, t_c)$ we have $\mu_c(t) < \mu_o$, then, the system possesses three sets of different solutions $(\bar{u}_i, \bar{p}_i(t), \bar{\varsigma}_i(t))$, $i = 1, 2, 3$. In the case that the three solutions $\varsigma_i(t)$ are in the following ordering

$$-a\mu_o \le \bar{\varsigma}_3(t) \le \bar{\varsigma}_2(t) \le 0 \le \bar{\varsigma}_1(t) \quad \forall t \in I_b, \tag{60}$$

then over the period I_b, the solution set $(\bar{u}_1(t), \bar{p}_1(t), \bar{\varsigma}_1(t))$ satisfies either (58) or (59); while the solution sets $(\bar{u}_i(t), \bar{p}_i(t), \bar{\varsigma}_i(t))$ $(i = 2, 3)$ satisfy

$$\Pi_\mu(\bar{u}_i) = \min_u \Pi_\mu(u) = \max_p \min_\varsigma \Pi^d_\mu(p, \varsigma) = \Pi^d_\mu(\bar{p}_i, \bar{\varsigma}_i) \quad i = 2, 3. \tag{61}$$

This theorem can be proved by combining the multi-solution theorem given by Gao (1999, Theorem 3.4.4) and the triality theorem.

Remark. By Theorem 3.4.4 proved by the author (Gao, 1999), for any given continuous function $\sigma(t)$, if $\bar{\varsigma}_i(t)$ $(i = 1, 2, 3)$ are the three solutions of the dual Euler-Lagrange equation (56) in the order of (60), then the associated $\bar{u}_1(t)$ is a global minimizer of the total potential

$$P_\mu(u) = \int_I \frac{1}{2} a (\frac{1}{2} u^2 - \mu_o)^2 \, \mathrm{d}t - \int_I \sigma(t) u \, \mathrm{d}t;$$

while $\bar{u}_2(t)$ is a local minimizer of P_μ and $\bar{u}_3(t)$ is a local maximizer of P_μ. ■

In *algebraic geometry*, the dual Euler-Lagrange equation (56) is the so-called *singular algebraic curve* in (ς, σ)-space, i.e. $\varsigma = 0$ is on the curve (see Silverman & Tate, 1992, p. 99). With a change of variables, the singular cubic curve (56) can be given by the well-known *Weierstrass equation*

$$y^2 = x^3 + \alpha x^2 + \beta x + \gamma,$$

where α, β, $\gamma \in \mathbb{R}$ are constants. If we let $\mathcal{C}_{ns}$ be a set consisting of non-singular points on the curve, then $\mathcal{C}_{ns}$ is an Abelian group. This fact in algebraic geometry is very important in understanding the stability of the nonconvex dynamical systems. Actually, from Figure 3 we can see clearly that for a given input control, if $\mu_c(t) < \mu_o$, the cubic algebraic equation (56) possesses three different real solutions for $\varsigma(t)$. The two negative solutions $\bar{\varsigma}(t)$ are the sources that lead to the chaotic motion of the system. Thus, the inequality (57) provides a *bifurcation (or chaotic) criterion* for the Duffing system. Fig. 3 also shows that if the continuous function $\sigma(t) = \mu(t) - p'(t)$ is one-signed on certain time interval $I_b \subset I = (0, t_c)$, each root $\bar{\varsigma}(t)$ of (56) is also one-signed on I_i.

Theoretically speaking, for the same initial conditions, the Duffing equation (52) and its dual system (55-56) should have the same solution set. Numerically, the primal and dual Duffing problems give complementary bounding approaches to the real solution. For the given data $a = 1, \mu_o = 1.5, u_0 = 2, v_0 = 1.4$ and $\mu = 0$, Figures 4 and 5 show the numerical primal (solid line) and dual (dashed line) solutions. From the dual trajectories

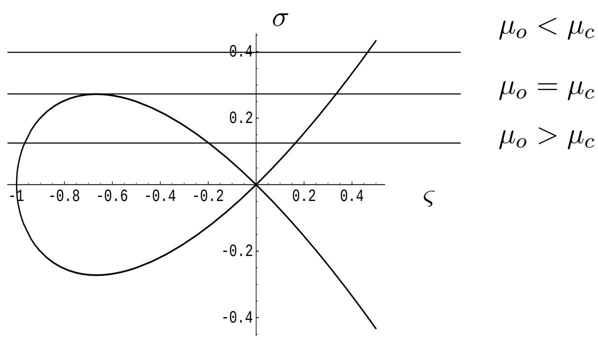


Figure 3: Singular algebraic curve for the dual Duffing equation (56)

in the dual phase space ς-p-$p_{,t}$ (Fig. 5(c-d)) we can see that at the point $\varsigma_3(t) = -a\mu_o$, if the function $\sigma(t) = \mu(t) - p_{,t}(t)$ changes its sign, the state $u(t)$ crosses the origin goes to another potential well in the phase space $\mathcal{Z} = \mathcal{U} \times \mathcal{V}^*$, and the bifurcation is then occurred. Thus, based on the canonical dual transformation method and theorems developed in this paper, the dual feedback control against the chaotic vibration of the Duffing system can be suggested as the following.

1. Periodic vibration on the whole phase plane.

Choosing the controller $\mu(t)$ such that the function $\sigma(t) = \mu(t) - p'(t)$ changes its sign at the point $\bar{\varsigma}_3(t) = -a\mu_o$.

2. Unilateral vibrations on half phase planes (either $u(t) > 0$ or $u(t) < 0$).

There are two methods: (1) choosing the controller $\mu(t)$ such that the function $\sigma(t) = \mu(t) - p'(t)$ does not change its sign at the point $\bar{\varsigma}_3(t) = -a\mu_o$; (2) choosing $\mu(t)$ such that either

$$\mu(t) > p'(t) + \left(a(2\mu_o/3)^3\right)^{1/2} \quad \forall t \in I, \tag{62}$$

or

$$\mu(t) < p'(t) - \left(a(2\mu_o/3)^3\right)^{1/2} \quad \forall t \in I. \tag{63}$$

Detailed study on the exact controllability and stability for the Duffing system will be given in other papers (cf. e.g., Gao, 2000d).

6 Concluding Remarks

The concept of duality is one of the most successful ideas in modern mathematics and science. The inner beauty of duality theory owes much to the fact that the nature was originally created in a duality way. By the fact that the canonical physical variables appear always in pairs, the canonical dual transformation method can be used to solve many problems in natural systems. The associated extended Lagrange duality and triality theories have profound computational impacts. For any given nonlinear problem, as long as there

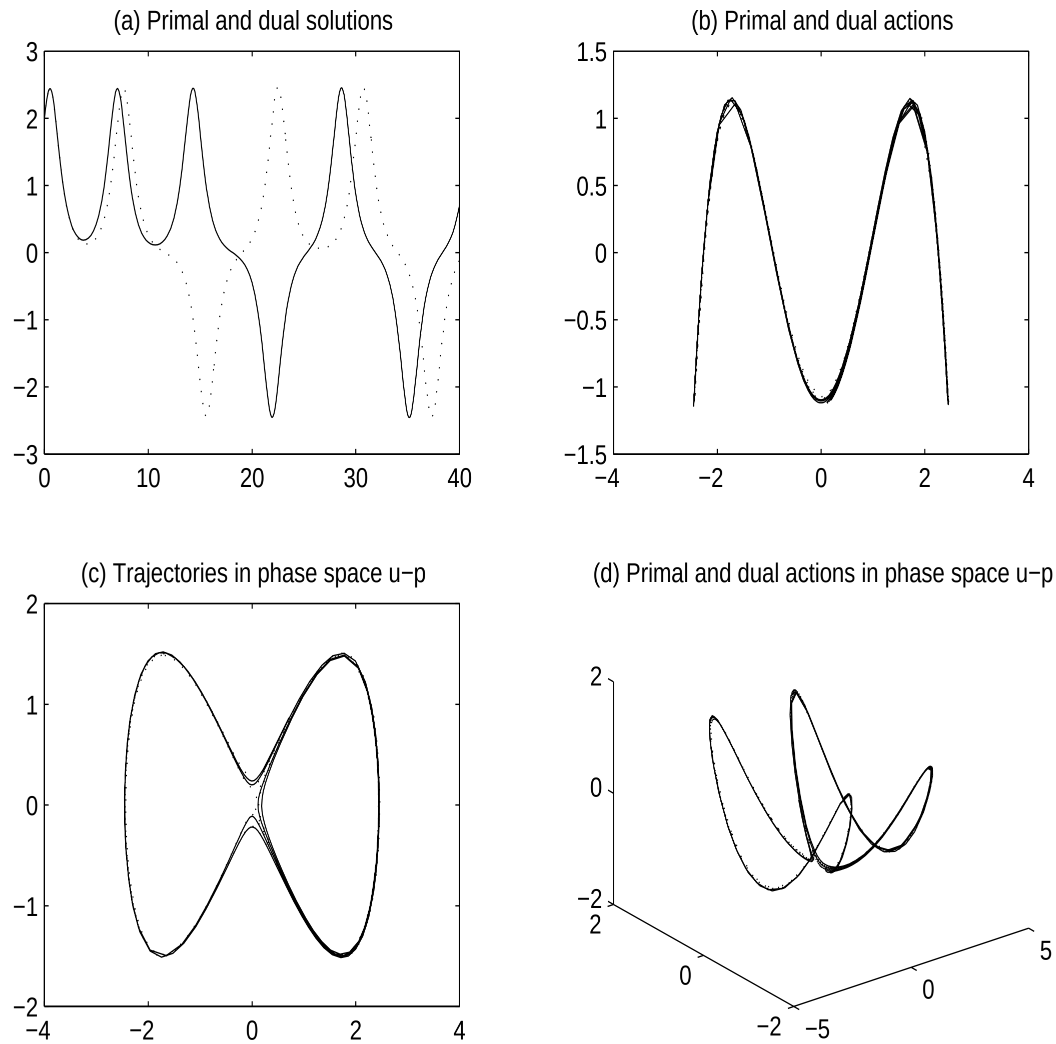


Figure 4: Primal and dual solutions in primal phase space

exists a geometrical operator Λ such that the tri-canonical forms can be characterized correctly, the canonical dual transformation method and the associated triality principles can be used to establish nice theories and to develop powerful alternative algorithms for robust feedback control of chaotic systems. For static three-dimensional finite deformation problems, a general analytic solution form and associated extremality theory have been proposed (Gao, 1999, 1999b). A general canonical dual transformation method for solving nonsmooth global optimization is given recently (Gao, 2000c). In general n-dimensional distributed parameter systems, the dual algebraic equation (56) will be a tensor equation and the stability of the nonconvex system will depend on the eigenvalues of symmetrical canonical stress tensor field $\varsigma(x, t)$ (see Gao, 2000d). The triality theory can be used for

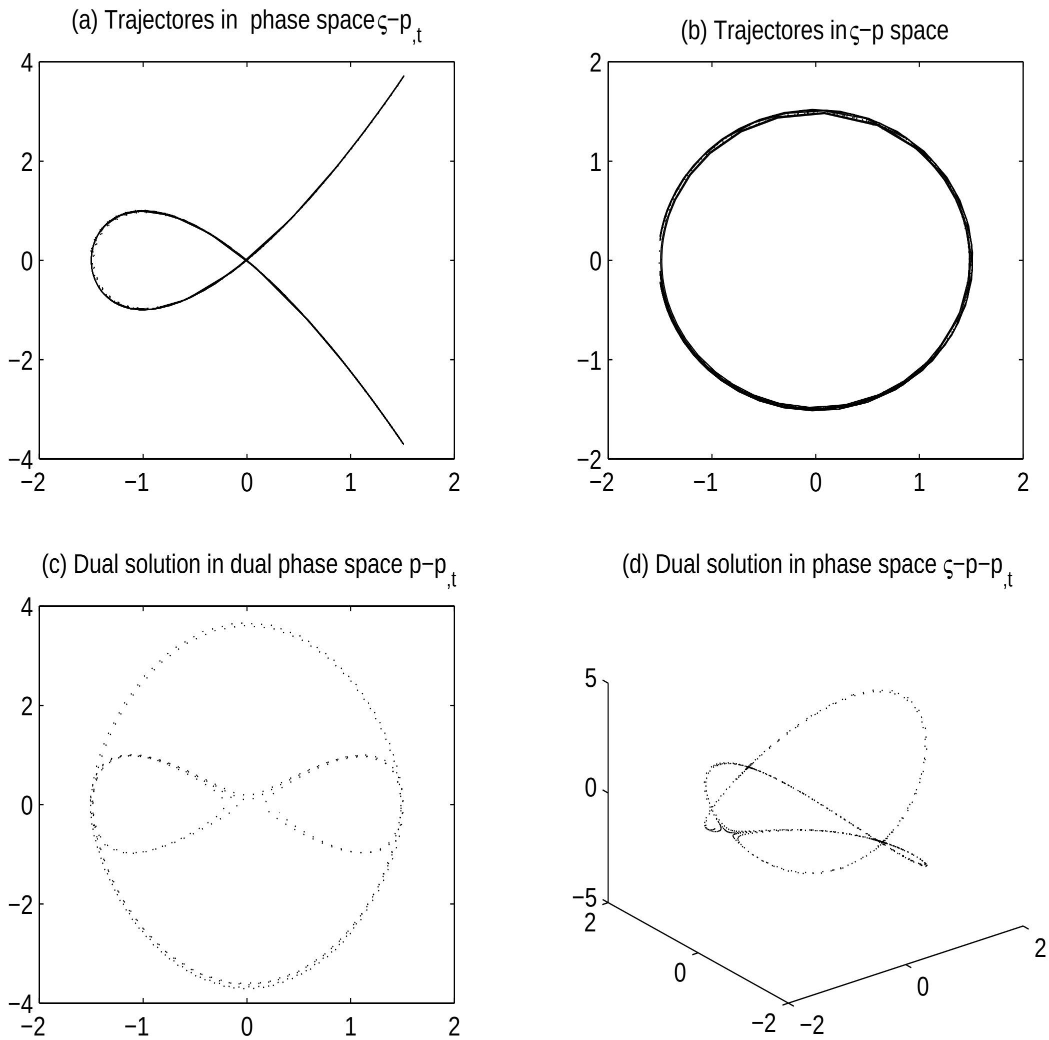


Figure 5: Duffing solutions in dual phase spaces

studying the controllability, observability and stability of distributed parameter control problems.

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