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Complementarity, polarity and triality in non-smooth, non-convex and non-conservative Hamilton systems

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Dedicated to Professor F. Pfeiffer on the occasion of his 65th birthday

This paper presents a unified critical-point theory in non-smooth, non-convex and dissipative Hamilton systems. The canonical dual/polar transformation methods and the associated bi-duality and triality theories proposed recently in non-convex variational problems are generalized into fully nonlinear dissipative dynamical systems governed by non-smooth constitutive laws and boundary conditions. It is shown that by this method, non-smooth and non-convex Hamilton systems can be reformulated into certain smooth dual, complementary and polar variational problems. Based on a newly proposed *polar Hamiltonian*, a nice bi-polarity variational principle is established for three-dimensional non-smooth elastodynamical systems, and a potentially powerful complementary variational principle can be used for solving unilateral variational inequality problems governed by non-smooth boundary conditions.

Keywords: non-smooth elastodynamics; non-convex variational problems; non-conservative Hamilton systems; complementarity; duality; polarity; triality; critical-point theory; contact mechanics

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1. Non-smooth and non-convex problems

The primary goal of this paper is to study the critical-point theory and extremality conditions of the general non-convex, non-smooth and non-conservative dynamical system governed by the stationary variational problem

$$(\mathcal{P}) : \mathit{\Pi}(u) = K(\partial_t u) - P(u) \to \mathrm{sta} \quad \forall u \in \mathcal{U}_k, \tag{1.1}$$

Author: does subscript 't' denote time (label 't' or variable 't')? How about subscript 'k'?

where the feasible space $\mathcal{U}_k$ is a convex, non-empty subset of a vector space $\mathcal{U}$ over an open space-time domain $\mathit{\Omega}_t = \mathit{\Omega} \times (0, t_c) \subset \mathbb{R}^n \times \mathbb{R}^+$, in which the essential boundary-initial conditions and certain constraints are prescribed: $\partial_t : \mathcal{U}_k \to \mathcal{V}$ is a time-differential operator from $\mathcal{U}_k$ to the space of velocity $\mathcal{V}$; $K(v)$ stands for the total kinetic energy of the system and $P : \mathcal{U}_k \to \mathbb{R}$ is the total potential of the system; the notation $\mathit{\Pi}(u) \to \mathrm{sta}\ \forall u \in \mathcal{U}_k$ stands for finding the stationary (or critical) points of $\mathit{\Pi}$ over the feasible space $\mathcal{U}_k$.

The general variational form of the problem $(\mathcal{P})$ covers a great variety of situations. In Newtonian mechanics, the operator ∂_t is simply the time derivative $\partial/\partial t$, the kinetic energy $K(v)$ is usually a convex (quadratic) differentiable functional, and the velocity-momentum relation $p = DK(v)$ is invertible. The complementary kinetic energy $K^*(p)$ can be obtained easily by the classical Legendre transformation. If the total potential $P(u)$ is convex, then the total action $\Pi(u)$ of the system is a so-called *DC functional*, i.e. the difference of convex functionals. In this case, the classical Hamiltonian $H(u,p) = K^*(p) + P(u)$ is a convex functional and the stationary condition of (1.1) leads to a particularly symmetrical structure in classical conservative dynamical systems (cf., for example, Marsden & Ratiu 1995)

$$u_{,t} = D_p H(u,p), \qquad p_{,t} = -D_u H(u,p).$$

However, this beautiful symmetry is broken in non-conservative finite deformation systems, where the total potential P is usually non-convex, or even non-smooth, which leads to many substantial difficulties in solving problem (1.1).

Non-smooth, non-convex and non-conservative phenomena arise naturally from real-life systems. Many problems in modern sciences (such as hysteresis and phase transitions, superconductivity, cosmology, mathematical economics, composite and smart materials, frictional contact problems and damage/fracture mechanics, nonlinear bifurcation and post-buckling analysis of large deformed structures, etc.) require the consideration of non-differentiability and non-convexity for their accurate mathematical modelling (see, for example, Moreau *et al*. 1988; Pfeiffer & Glocker 1996; Dem'yanov *et al*. 1996; Brokate & Sprekels 1996; Kibble 1997; Mistakidis & Stavroulakis 1998; Motreanu & Panagiotopoulos 1999; Gao 2000*a*–*d*; Gao *et al*. 2001, among many others). Generally speaking, due to the non-differentiability of the non-smooth total potential, traditional direct methods for solving non-smooth and non-convex variational problems are usually very difficult, or even impossible. Mathematically speaking, the chaotic phenomenon of a nonlinear dynamical system is mainly due to the non-convexity of the total potential of the system. Very small perturbations of the system's initial conditions and parameters may lead the system to different potential wells with significantly different performance characteristics. The numerical results vary with the methods used (cf., for example, Gao 2000*a*). This is one of the main reasons why the traditional perturbation analysis and the direct approaches cannot successfully be applied to chaotic systems.

Author: '*a*–*d*' added to 2000 and '*a*' removed from 2001 – OK?

Duality theory plays fundamental roles in natural phenomena. In engineering mechanics, the study on the complementary-dual variational principles has a long history (cf., for example, Hellinger 1914; Reissner 1955; Oden & Reddy 1983; Ekeland 1990; Tabarrok & Rimrott 1994). During the last decade, the so-called *primal-dual interior point method* has emerged as the most important and efficient revolutionary technique in mathematical programming (cf., for example, Wright 1998). The advantage of the primal-dual approaches relying on a common mathematical structure that underlies many physical theories. A self-contained comprehensive presentation of the mathematical duality theory in general non-convex, non-smooth systems was given recently by Gao (2000*a*). It was shown that any non-smooth convex function can be easily converted into a smooth conjugate (dual) function by the classical Legendre transformation. A very interesting triality theory was established for non-convex systems.

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The aim of this article is to generalize my previous results on non-convex static problems into non-smooth, non-convex dissipative dynamical systems. The rest of this paper is divided into five main sections. The next section set up the notation used in the paper and describes the problems. A general framework and the canonical dual

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transformation method in fully nonlinear, non-smooth, non-conservative dynamical systems are discussed. Section 3 presents an extended Lagrangian duality theory in general non-convex Hamilton systems, by which the dual action can be obtained. The critical points in fully nonlinear systems are classified. In § 4, the interesting triality theory is generalized into canonical dynamical systems with general geometrically nonlinear operator Λ. Section 5 and 6 are devoted mainly to the super-Lagrange duality theory in geometrically linear systems. The nice bi-polarity theory is proposed for non-smooth elastodynamics and a complementary action is presented for solving non-conservative dynamical system governed by non-smooth boundary conditions. Some concluding remarks are made in the last section.

2. Framework

Let $\mathcal{U}$ and $\mathcal{U}^*$ be two real linear spaces, finite- or infinite dimensional, placed in duality by a bilinear form $\langle\cdot,\cdot\rangle : \mathcal{U}\times\mathcal{U}^* \to \mathbb{R}$. For a given extended real-valued function $P : \mathcal{U} \to \bar{\mathbb{R}} = \mathbb{R}\cup\{-\infty\}\cup\{+\infty\}$, the sub- and super-differentials of P at $\bar{u}\in\mathcal{U}$ are defined by

$$\partial^- P(\bar{u}) = \{\bar{u}^* \in \mathcal{U}^* \mid P(u) - P(\bar{u}) \geqslant (\bar{u}^*, u-\bar{u})\ \forall u \in \mathcal{U}\},$$
$$\partial^+ P(\bar{u}) = \{\bar{u}^* \in \mathcal{U}^* \mid P(u) - P(\bar{u}) \leqslant (\bar{u}^*, u-\bar{u})\ \forall u \in \mathcal{U}\},$$

respectively. Clearly, we always have $\partial^+ P = -\partial^-(-P)$. In convex analysis, it is convention that ∂^- is simply written as ∂. In this paper, ∂ stands for either ∂^- or ∂^+, i.e.

$$\partial = \{\partial^-, \partial^+\}.$$

If P is finite and Gâteaux differentiable on a subspace $\mathcal{U}_{\rm s} \subset \mathcal{U}$, then

Author: what does subscript 's' denote?

$$\partial P(\bar{u}) = \partial^- P(\bar{u}) = \partial^+ P(\bar{u}) = \{DP(\bar{u})\},$$

where $DP : \mathcal{U}_{\rm s} \to \mathcal{U}^*$ denotes the Gâteaux derivative of P at $\bar{u}$. The following notation and definitions, used in Gao (2000*a*), will be of convenience in non-convex, non-smooth variational analysis.

Definition 2.1. The set of functions $P : \mathcal{U} \to \bar{\mathbb{R}}$ which are either convex or concave is denoted by $\Gamma(\mathcal{U})$. In particular, let $\check{\Gamma}(\mathcal{U})$ denote the subset of functions $P \in \Gamma(\mathcal{U})$ which are convex and $\hat{\Gamma}(\mathcal{U})$ the subset of $P \in \Gamma(\mathcal{U})$ which are concave.

(D1) The *canonical function space* $\Gamma_{\rm G}(\mathcal{U}_{\rm s})$ is a subset of functions $P \in \Gamma(\mathcal{U}_{\rm s})$ which are Gâteaux differentiable on $\mathcal{U}_{\rm s} \subset \mathcal{U}$.

(D2) The *extended canonical function space* $\Gamma_0(\mathcal{U})$ is a subset of functions $P \in \Gamma(\mathcal{U})$ which are either convex, lower semicontinuous or concave, upper semicontinuous, and if P takes the values $\pm\infty$, then P is identically equal to $\pm\infty$.

By the *Fenchel transformation*, the *super-conjugate function* of an extended function $P : \mathcal{U} \to \bar{\mathbb{R}}$ is defined by

$$P^\sharp(u^*) = \sup_{u\in\mathcal{U}}\{\langle u, u^*\rangle - P(u)\}.$$

From the theory of convex analysis we know that $P^\sharp : \mathcal{U}^* \to \bar{\mathbb{R}}$ is always convex and lower semicontinuous, i.e. $P^\sharp \in \check{\Gamma}_0(\mathcal{U}^*)$. Dually, the *sub-conjugate function* of P, defined by

$$P^\flat(u^*) = \inf_{u\in\mathcal{U}}\{\langle u, u^*\rangle - P(u)\},$$

is always concave and upper semicontinuous, i.e. $P^\flat \in \hat{\Gamma}_0(\mathcal{U}^*)$, and $P^\flat = -P^\sharp$. Both the super- and sub-conjugates are called Fenchel conjugate functions and we write $P^* = \{P^\flat, P^\sharp\}$. Thus the extended Fenchel transformation can be written as

$$P^*(u^*) = \operatorname{ext}\{\langle u, u^*\rangle - P(u) \mid u \in \mathcal{U}\}. \tag{2.1}$$

Clearly, if $P \in \Gamma_0(\mathcal{U})$, we have the *Fenchel duality relations*, namely

$$u^* \in \partial P(u) \quad \Leftrightarrow \quad u \in \partial P^*(u^*) \quad \Leftrightarrow \quad P(u) + P^*(u^*) = \langle u, u^*\rangle. \tag{2.2}$$

The pair (u, u^*) is called the *Fenchel duality pair* on $\mathcal{U} \times \mathcal{U}^*$ if and only if equation (2.2) holds on $\mathcal{U} \times \mathcal{U}^*$.

The conjugate pair (u, u^*) is called the *Legendre duality pair* on $\mathcal{U}_\mathrm{s} \times \mathcal{U}_\mathrm{s}^* \subset \mathcal{U} \times \mathcal{U}^*$ if and only if the equivalent relations

$$u^* = DP(u) \quad \Leftrightarrow \quad u = DP^*(u^*) \quad \Leftrightarrow \quad P(u) + P^*(u^*) = \langle u, u^*\rangle. \tag{2.3}$$

hold on $\mathcal{U}_\mathrm{s} \times \mathcal{U}_\mathrm{s}^*$.

In non-convex Hamilton systems, the total potential $P : \mathcal{U} \to \bar{\mathbb{R}}$ is usually non-convex. In order to study duality theory, we introduce a pair of the so-called *intermediate variable spaces* $(\mathcal{E}, \mathcal{E}^*)$, placed in duality by the second bilinear form $\langle\cdot;\cdot\rangle : \mathcal{E}\times\mathcal{E}^* \to \mathbb{R}$. We assume that there exists a Gâteaux-differentiable operator $\Lambda : \mathcal{U} \to \mathcal{E}$ and canonical functionals $F \in \Gamma_0(\mathcal{U})$, $E \in \Gamma_0(\mathcal{E})$ such that

$$P(u) = E(\Lambda(u)) - F(u). \tag{2.4}$$

In finite deformation theory, Λ is usually a gradient-like operator and $E \in \Gamma_0(\mathcal{E})$ is called the stored potential energy. The element $\xi = \Lambda(u) \in \mathcal{E}$ is called canonical strain measure (see Gao 2000*a*) and the duality relation $\xi^* \in \partial_\xi J_\xi(u, \xi) = \partial E(\xi)$ is called the *constitutive law*. The functional $F \in \Gamma_0(\mathcal{U})$ stands for the external energy, and the duality relation $u^* \in \partial F(u)$ leads to the boundary conditions (see Gao 2000*a*). Let $\mathcal{U}_\mathrm{a} \subset \mathcal{U}$ and $\mathcal{E}_\mathrm{a} \subset \mathcal{E}$ be two subsets on which the extended canonical functionals $F \in \Gamma_\mathrm{G}(\mathcal{U}_\mathrm{a})$, $E \in \Gamma_\mathrm{G}(\mathcal{E}_\mathrm{a})$ are Gâteaux differentiable. Let $\mathcal{U}_\mathrm{a}^* \subset \mathcal{U}^*$ and $\mathcal{E}_\mathrm{a}^* \subset \mathcal{E}^*$ be the ranges of the duality mappings $DF : \mathcal{U}_\mathrm{a} \to \mathcal{U}^*$ and $DE : \mathcal{E}_\mathrm{a} \to \mathcal{E}^*$, respectively. Then on $(\mathcal{U}_\mathrm{a}, \mathcal{U}_\mathrm{a}^*)$ and $(\mathcal{E}_\mathrm{a}, \mathcal{E}_\mathrm{a}^*)$, we have the Legendre duality relations

Author: what does subscript 'a' denote? Please check that all subscripts and superscripts ('k', 'G', etc.) that are labels are roman (e.g. does 'c' denote canonical?) and those that refer to variables are italic. Please also clarify use of lower case 'o' in sub- and superscripts (i.e. should they be zeros).

$$\begin{aligned} u^* = DF(u) \quad &\Leftrightarrow \quad u = DF^*(u^*) \quad \Leftrightarrow \quad F(u) + F^*(u) = \langle u, u^*\rangle, \\ \xi^* = DE(\xi) \quad &\Leftrightarrow \quad \xi = DE^*(\xi^*) \quad \Leftrightarrow \quad E(\xi) + E^*(\xi^*) = \langle \xi; \xi^*\rangle. \end{aligned}$$

The directional derivative of ξ at $\bar{u}$ in the direction $u \in \mathcal{U}$ is defined by

$$\delta\xi(\bar{u}; u) := \lim_{\theta\to 0^+} \frac{\xi(\bar{u} + \theta u) - \xi(\bar{u})}{\theta} = \Lambda_t(\bar{u})u, \tag{2.5}$$

where $\Lambda_t(\bar{u}) = D\Lambda(\bar{u})$ denotes the Gâteaux derivative of the operator Λ at $\bar{u}$. For a given $\xi^* \in \mathcal{E}^*$, $G_{\xi^*}(u) = \langle \Lambda(u); \xi^*\rangle$ is a real-valued function of u on $\mathcal{U}$. Its Gâteaux derivative at $\bar{u} \in \mathcal{U}_\mathrm{a}$ in the direction $u \in \mathcal{U}$ reads

$$\delta G_{\xi^*}(\bar{u}; u) = \langle \Lambda_t(\bar{u})u; \xi^*\rangle = \langle u, \Lambda_t^*(\bar{u})\xi^*\rangle,$$

where $\Lambda_t^*(\bar{u}) : \mathcal{E}^* \to \mathcal{U}^*$ is the adjoint operator of Λ_t associated with the two bilinear forms. The *complementary operator* $\Lambda_c(u)$ is defined by the geometrical operator decomposition (Gao & Strang 1989):

$$\Lambda(u) = \Lambda_t(u)u + \Lambda_c(u).$$

We have

$$G_{\xi^*}(u) = \langle \Lambda(u); \xi^* \rangle = \langle u, \Lambda_t^*(u)\xi^* \rangle - G(u, \xi^*), \tag{2.6}$$

where $G(u, \xi^*) = \langle -\Lambda_c(u); \xi^* \rangle$ is the so-called *complementary gap functional*, introduced by Gao & Strang (1989). This gap function plays an essential role in non-convex variational analysis.

For dynamical problems, we let $\mathcal{V}$ and $\mathcal{V}^*$ be the velocity and momentum space, respectively, placed in duality by the third bilinear form $\langle *, * \rangle : \mathcal{V} \times \mathcal{V}^* \to \mathbb{R}$. Let $\mathcal{V}_a \subset \mathcal{V}$ be a subspace in which certain essential time initial/boundary conditions are given. Dually, let $\mathcal{V}_a^*$ be a subset of $\mathcal{V}^*$, in which the complementary (natural) conditions are given. Thus, for a given time-differential operator $\partial_t : \mathcal{U}_a \to \mathcal{V}_a$, the adjoint $\partial_t^* : \mathcal{V}_a^* \to \mathcal{U}^*$ can be defined by

$$\langle \partial_t u, p \rangle = \langle u, \partial_t^* p \rangle \quad \forall u \in \mathcal{U}_a, \quad \forall p \in \mathcal{V}_a^*.$$

For Newtonian systems, the kinetic energy $K : \mathcal{V} \to \mathbb{R}$ and its Legendre conjugate $K^* : \mathcal{V}_a^* \to \mathbb{R}$ are usually quadratic functionals:

$$K(v) = \tfrac{1}{2}\langle v, \rho v \rangle, \qquad K^*(p) = \tfrac{1}{2}\langle \rho^{-1} p, p \rangle.$$

Thus, on $\mathcal{V}_a \times \mathcal{V}_a^*$, the canonical physical relations between the duality pair (v, p) are linear, i.e.

$$p = DK(v) = \rho v \quad \Leftrightarrow \quad v = DK^*(p) = \rho^{-1} p.$$

Let $\Phi(u, v, \xi) = K(v) - E(\xi) + F(u) : \mathcal{U} \times \mathcal{V} \times \mathcal{E} \to \bar{\mathbb{R}}$ be the extended canonical functional with effective domain $\operatorname{dom} \Phi = \mathcal{U}_a \times \mathcal{V}_a \times \mathcal{E}_a$ such that $\Pi(u) = \Phi(u, \partial_t u, \Lambda(u))$. Then the *kinematically admissible space* $\mathcal{U}_k$ can be written in a standard canonical form:

$$\mathcal{U}_k = \{ u \in \mathcal{U}_a \mid \Lambda(u) \in \mathcal{E}_a,\ \partial_t u \in \mathcal{V}_a \}. \tag{2.7}$$

In terms of the canonical variables (u, v, ξ), the primal problem $(\mathcal{P})$ can be reformulated in the canonical form as follows.

Problem 2.2 (canonical primal problem). *Suppose that $\Phi : \mathcal{U}_a \times \mathcal{V}_a \times \mathcal{E}_a \to \mathbb{R}$ is a canonical functional such that $\mathcal{U}_k$ is not empty and $\Pi(u) = \Phi(u, \partial_t u, \Lambda(u))$ for each $u \in \mathcal{U}_k$. Find the critical point of Π such that*

$$(\mathcal{P}_c) : \Pi(u) = \Phi(u, \partial_t u, \Lambda(u)) \to \text{sta} \quad \forall u \in \mathcal{U}_k. \tag{2.8}$$

The following classification for dynamical systems was originally introduced in nonlinear variational/boundary-value problems by Gao (1998*a*, *b*, 2000*a*). Author: *a*, *b* added – OK?

Definition 2.3. Suppose that for a given problem $(\mathcal{P})$, the total action $\Pi(u)$ is well defined on its domain $\mathcal{U}_k \subset \mathcal{U}$. If the generalized geometrical operator $\Lambda : \mathcal{U} \to \mathcal{E}$ can be chosen in such a way that $\Pi(u) = \Phi(u, \partial_t u, \Lambda(u))$, and $\Phi \in \Gamma_G(\mathcal{U}_a) \times \Gamma_G(\mathcal{V}_a) \times \Gamma_G(\mathcal{E}_a)$, then

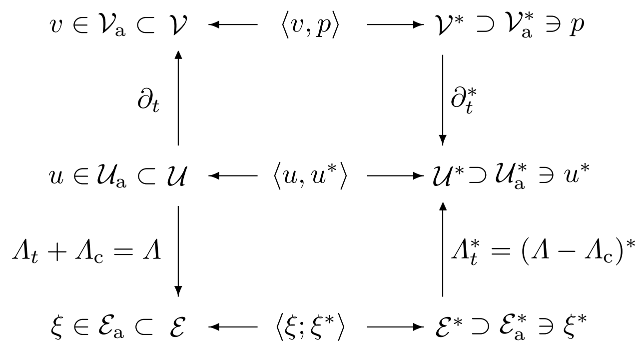

Figure 1. Framework in fully nonlinear Newtonian systems.

(1) the transformation $\{\Pi; \mathcal{U}_k\} \to \{\Phi; \mathcal{U}_a \times \mathcal{V}_a \times \mathcal{E}_a\}$ is called the *canonical transformation*, and $\Phi : \mathcal{U}_a \times \mathcal{V}_a \times \mathcal{E}_a \to \mathbb{R}$ is called the *canonical action associated with* Λ; and

(2) the problem $(\mathcal{P})$ is called *geometrically nonlinear (respectively linear)* if $\Lambda : \mathcal{U} \to \mathcal{E}$ is nonlinear (respectively linear); it is called *physically nonlinear* (respectively linear) if the duality mapping $D\Phi : \mathcal{U}_a \times \mathcal{V}_a \times \mathcal{E}_a \to \mathcal{U}_a^* \times \mathcal{V}_a^* \times \mathcal{E}_a^*$ is nonlinear (respectively linear); it is called *fully nonlinear* if it is both geometrically and physically nonlinear.

The canonical transformation plays a fundamental role in duality theory of nonconvex systems. Clearly, if $\Phi \in \Gamma_G(\mathcal{U}_a) \times \Gamma_G(\mathcal{V}_a) \times \Gamma_G(\mathcal{E}_a)$ is a canonical functional, the Gâteaux derivative $D\Phi : \mathcal{U}_a \times \mathcal{V}_a \times \mathcal{E}_a \to \mathcal{U}_a^* \times \mathcal{V}_a^* \times \mathcal{E}_a \subset \mathcal{U}^* \times \mathcal{V}^* \times \mathcal{E}^*$ is a monotone mapping, i.e. the duality relations

$$u^* = D_u\Phi(u, v, \xi), \qquad p = D_v\Phi(u, v, \xi), \qquad \xi^* = D_\xi\Phi(u, v, \xi) \tag{2.9}$$

are reversible on $\mathcal{U}_a \times \mathcal{V}_a \times \mathcal{E}_a$. Thus, on $\mathcal{U}_k$, the criticality condition

$$\begin{aligned}\delta\Pi(\bar{u}; u) &= \langle u, DF(\bar{u})\rangle + \langle \partial_t u, DK(\partial_t \bar{u})\rangle - \langle \Lambda_t(\bar{u})u; DE(\Lambda(\bar{u}))\rangle \\ &= \langle u, \bar{u}^*\rangle + \langle u, \partial_t^* \bar{p}\rangle - \langle u; \Lambda_t^*(\bar{u})\bar{\xi}^*\rangle = 0 \quad \forall u \in \mathcal{U}_k\end{aligned}$$

leads to the *fundamental inclusion*

$$0 \in \partial_t^* \partial K(\partial_t u) - \Lambda_t^*(u)\partial E(\Lambda(u)) + \partial F(u). \tag{2.10}$$

In terms of canonical variables, this fundamental inclusion can be written in the *trio-canonical forms*, namely

$$\left.\begin{array}{lll} (1) \text{ geometrical equations:} & v = \partial_t u, \quad \xi = \Lambda(u), & \\ (2) \text{ physical relations:} & p = DK(v), \quad \xi^* \in \partial E(\xi), \quad u^* \in \partial F(u), & \\ (3) \text{ balance equation:} & u^* = \Lambda_t^*(u)\xi^* - \partial_t^* p. & \end{array}\right\} \tag{2.11}$$

The framework for the fully nonlinear system is shown in figure 1. Extensive illustrations of the canonical transformation and the trio-canonical forms in mathematical physics and variational analysis were given in the monograph by Gao (2000*a*).

3. Canonical Hamiltonian, Lagrangian and dual action

In this section, we shall study extended Lagrange forms and the associated dual actions in fully nonlinear, non-smooth Hamilton systems with dissipation.

Definition 3.1. Let $\mathcal{Z} = \mathcal{U} \times \mathcal{V}^* \times \mathcal{E}^*$ be the so-called *extended canonical phase space.*

(D1) The functional $\Theta : \mathcal{Z} \to \bar{\mathbb{R}}$ defined by

$$\Theta(u, p, \xi^*) = K^*(p) - E^*(\xi^*) - F(u) \in \Gamma(\mathcal{U}) \times \Gamma(\mathcal{V}^*) \times \Gamma(\mathcal{E}^*) \tag{3.1}$$

is called the *extended canonical Hamiltonian density* associated with Π.

(D2) The functional $\Xi : \mathcal{Z} \to \bar{\mathbb{R}}$ defined by

$$\begin{aligned}\Xi(u, p, \xi^*) &= \langle \partial_t u, p \rangle - \langle \Lambda(u); \xi^* \rangle - \Theta(u, p, \xi^*) \\ &= \langle \partial_t u, p \rangle - \langle \Lambda(u); \xi^* \rangle - K^*(p) + E^*(\xi^*) + F(u)\end{aligned} \tag{3.2}$$

is called the *extended Lagrangian* of $(\mathcal{P})$ associated with Λ. It is called the *extended canonical Lagrangian form* if $\Xi \in \Gamma_0(\mathcal{U}) \times \Gamma_0(\mathcal{V}^*) \times \Gamma_0(\mathcal{E}^*)$.

Since $K^* \in \Gamma_0(\mathcal{V}^*)$, $E^* \in \Gamma_0(\mathcal{E}^*)$ are canonical functionals, it is easy to show that the fundamental form (2.10) is equivalent to the *extended canonical Hamilton forms*:

$$\left.\begin{aligned} \partial_t u &\in \partial_p \Theta(u, p, \xi^*), \\ \partial_t^* p - \Lambda_t^*(u)\xi^* &\in \partial_u \Theta(u, p, \xi^*), \\ -\Lambda(u) &\in \partial_{\xi^*} \Theta(u, p, \xi^*). \end{aligned}\right\} \tag{3.3}$$

In the case where $K \in \Gamma_{\mathrm{G}}(\mathcal{V}_{\mathrm{a}})$, $F \in \Gamma_{\mathrm{G}}(\mathcal{U}_{\mathrm{a}})$ and $E \in \Gamma_{\mathrm{G}}(\mathcal{E}_{\mathrm{a}})$ are Gâteaux-differentiable canonical functionals, then the extended Lagrangian Ξ is Gâteaux differentiable on $\mathcal{Z}_{\mathrm{a}} = \mathcal{U}_{\mathrm{a}} \times \mathcal{V}_{\mathrm{a}}^* \times \mathcal{E}_{\mathrm{a}}^*$, and the criticality condition $D\Xi(\bar{u}, \bar{p}, \bar{\xi}^*) = 0$ leads to the following *extended Lagrange equations*:

$$\left.\begin{aligned} \partial_t u &= DK^*(p), \\ \partial_t^* p &= \Lambda_t^*(u)\xi^* - DF(u), \\ \Lambda(u) &= DE^*(\xi^*). \end{aligned}\right\} \tag{3.4}$$

Since K, E and F are canonical functionals, we know that, by the Legendre duality theory, any critical point of Ξ solves the variational problem $(\mathcal{P})$. Thus, the canonical primal problem $(\mathcal{P}_{\mathrm{c}})$ is equivalent to the general variational problem on $\mathcal{Z}_{\mathrm{a}}$:

$$(\Xi) : \Xi(u, p, \xi^*) \to \mathrm{sta} \quad \forall (u, p, \xi^*) \in \mathcal{Z}_{\mathrm{a}}. \tag{3.5}$$

By the fact that $E(\xi) = \mathrm{ext}\{\langle \xi; \xi^* \rangle - E^*(\xi^*) \mid \forall \xi^* \in \mathcal{E}^*\}$, then, for any fixed $(u, p) \in \mathcal{U} \times \mathcal{V}$, we have

$$L(u, p) = \mathrm{ext}\{\Xi(u, p, \xi^*) \mid \forall \xi^* \in \mathcal{E}^*\} = \langle \partial_t u, p \rangle - K^*(p) - P(u),$$

which is the well-known classical Lagrangian.

In general, for a given $u \in \mathcal{U}$, the extended Lagrangian $\Xi : \mathcal{V}^* \times \mathcal{E}^* \to \bar{\mathbb{R}}$ is an extended canonical functional, and

$$\Pi(u) = \mathrm{sta}\{\Xi(u, p, \xi^*) \mid \forall (p, \xi^*) \in \mathcal{V}^* \times \mathcal{E}^*\}. \tag{3.6}$$

Particularly, if $E \in \check{\Gamma}_{\rm G}(\mathcal{E}_{\rm a})$, $K \in \check{\Gamma}_{\rm G}(\mathcal{V}_{\rm a})$, then $\Xi : \mathcal{V}_{\rm a}^* \times \mathcal{E}_{\rm a}^* \to \mathbb{R}$ is a saddle functional for any given $u \in \mathcal{U}_{\rm a}$. In this case, the stationary problem (3.6) is actually the saddle minimax problem

$$\Pi(u) = \inf_{\xi^* \in \mathcal{E}_{\rm a}^*} \sup_{p \in \mathcal{V}_{\rm a}^*} \Xi(u, p, \xi^*) = \sup_{p \in \mathcal{V}_{\rm a}^*} \inf_{\xi^* \in \mathcal{E}_{\rm a}^*} \Xi(u, p, \xi^*) \quad \forall u \in \mathcal{U}_{\rm a}. \tag{3.7}$$

On the other hand, for any given $(p, \xi^*) \in \mathcal{V}_{\rm a}^* \times \mathcal{E}_{\rm a}^*$, the convexity of $\Xi(u, \cdot, \cdot) \to \mathbb{R}$ depends on the operator Λ and $\xi^* \in \mathcal{E}^*$. The dual action $\Pi^{\rm d} : \mathcal{V}_{\rm a}^* \times \mathcal{E}_{\rm a}^* \to \mathbb{R}$ can be defined by

$$\begin{aligned} \Pi^{\rm d}(p, \xi^*) &= \text{sta}\{\Xi(u, p, \xi^*) \mid \forall u \in \mathcal{U}_{\rm a}\} \\ &= E^*(\xi^*) - K^*(p) - F^{\Lambda}(p, \xi^*) \quad \forall (p, \xi^*) \in \mathcal{V}_{\rm a}^* \times \mathcal{E}_{\rm a}^*, \end{aligned} \tag{3.8}$$

where $F^{\Lambda}(p, \xi^*)$ is the so-called Λ-dual functional of $F(u)$ defined by the *canonical dual transformation*

$$F^{\Lambda}(p, \xi^*) = \text{sta}\{\langle \Lambda(u); \xi^* \rangle - \langle \partial_t u, p \rangle - F(u) \mid \forall u \in \mathcal{U}\}. \tag{3.9}$$

For any given $(p, \xi^*) \in \mathcal{V}^* \times \mathcal{E}^*$, the solution $\bar{u}$ of this stationary variational problem satisfying the balance equation: Clarify sentence?

$$\Lambda_t^*(\bar{u})\xi^* - \partial_t^* p \in \partial F(\bar{u}). \tag{3.10}$$

Let $\mathcal{W}_{\rm s}^* \subset \mathcal{V}_{\rm a}^* \times \mathcal{E}_{\rm a}^*$ be the so-called *dual feasible space* such that for any given $(p, \xi^*) \in \mathcal{W}_{\rm s}^*$, the critical point $\bar{u}$ of the stationary problem (3.9) can be determined as $\bar{u} = \bar{u}(p, \xi^*)$, the problem that is dual to the canonical primal problem $(\mathcal{P})$ can be proposed as the following.

Problem 3.2 (canonical dual variational problem). *Suppose that the dual feasible space $\mathcal{W}_{\rm s}^*$ is not empty. Find the critical point of $\Pi^{\rm d}$ such that*

$$(\mathcal{P}^{\rm d}) : \Pi^{\rm d}(p, \xi^*) \to \text{sta} \quad \forall (p, \xi^*) \in \mathcal{W}_{\rm s}^*. \tag{3.11}$$

The following lemma plays a key role in duality theory for nonlinear dynamical systems.

Lemma 3.3. *Let $\Xi(u, p, \xi^*)$ be a given extended Lagrangian associated with $(\mathcal{P}_{\rm c})$ and $\Pi^{\rm d}(p, \xi^*)$ the dual action defined by (3.8). Suppose that $\mathcal{Z}_{\rm r} = \mathcal{U}_{\rm r} \times \mathcal{V}_{\rm r}^* \times \mathcal{E}_{\rm r}^*$ is an open subset of $\mathcal{Z}_{\rm a}$ and $(\bar{u}, \bar{p}, \bar{\xi}^*) \in \mathcal{Z}_{\rm r}$ is a critical point of Ξ on $\mathcal{Z}_{\rm r}$, Π is Gâteaux differentiable at $\bar{u}$, and $\Pi^{\rm d}$ is Gâteaux differentiable at $(\bar{p}, \bar{\xi}^*)$. Then $D\Pi(\bar{u}) = 0$, $D\Pi^{\rm d}(\bar{p}, \bar{\xi}^*) = 0$, and*

$$\Pi(\bar{u}) = \Xi(\bar{u}, \bar{p}, \bar{\xi}^*) = \Pi^{\rm d}(\bar{p}, \bar{\xi}^*). \tag{3.12}$$

This lemma can be proved by using some fundamental results in calculus of variations (cf., for example, Gao 2000*a*). Lemma 3.3 shows that the critical points of the extended Lagrangian are also the critical points for both the primal and dual variational problems. The extremality of the primal and dual actions will be discussed in the next section.

In the case that $F(u)$ is a linear functional, then by the Riesz representation theory we know that there exists an element $\bar{u}^* \in \mathcal{U}^*$ such that $F(u) = \langle u, \bar{u}^* \rangle$. Thus, $DF(\bar{u}) = \bar{u}^*$ and the critical point $\bar{u}$ of problem (3.9) satisfy the balance equation

$$\partial_t^* p - \Lambda_t^*(\bar{u})\xi^* + \bar{u}^* = 0. \tag{3.13}$$

If for any given $(p,\xi^*)\in\mathcal{W}_s^*$ the critical point $\bar{u}=\bar{u}(p,\xi^*)$ can be well determined by this balance equation, then by the canonical operator decomposition $\Lambda=\Lambda_t+\Lambda_c$, we have

$$F^{\Lambda}(p,\xi^*)=-G^{\mathrm{d}}(p,\xi^*),\quad \text{s.t. } (p,\xi^*)\in\mathcal{W}_s^*, \tag{3.14}$$

Author: what is 's.t.'?

where $G^{\mathrm{d}}(p,\xi^*)=\langle-\Lambda_c(\bar{u}(p,\xi^*));\xi^*\rangle=G(\bar{u},\xi^*)$ is the *pure complementary gap function* (see Gao 2000*a*). In this case, the dual action reads

$$\Pi^{\mathrm{d}}(p,\xi^*)=E^*(\xi^*)+G^{\mathrm{d}}(p,\xi^*)-K^*(p). \tag{3.15}$$

Clearly, $G^{\mathrm{d}}(p,\xi^*)=0$ if Λ is a linear operator. In this case, $\Xi(u,p,\xi^*)$ is a canonical functional and Π^{d} can be considered as the Fenchel–Rockafellar dual action. Lemma 3.3 shows that in non-convex systems, the duality gap existing in the Fenchel–Rockafellar duality theory is recovered by the complementary gap functional.

4. Triality theory in fully nonlinear dynamical systems

In order to clarify the extremality conditions of the extended Lagrangian, in this section we shall assume that $E\in\check{\Gamma}_G(\mathcal{E}_a)$, $K\in\check{\Gamma}_G(\mathcal{V}_a)$ are convex canonical functionals and that $F(u)=\langle u,\bar{u}^*\rangle$ is a linear functional on $\mathcal{U}_a$. Let $\mathcal{Z}_c\subset\mathcal{Z}_a=\mathcal{U}_a\times\mathcal{V}_a^*\times\mathcal{E}_a^*$ be a critical point set of Ξ, i.e.

$$\mathcal{Z}_c=\{(\bar{u},\bar{p},\bar{\xi}^*)\in\mathcal{Z}_a \mid \delta\Xi(\bar{u},\bar{p},\bar{\xi}^*;u,p,\xi^*)=0\ \forall(u,p,\xi^*)\in\mathcal{Z}_a\}.$$

For any given critical point $(\bar{u},\bar{p},\bar{\xi}^*)\in\mathcal{Z}_c$, we let $\mathcal{Z}_r=\mathcal{U}_r\times\mathcal{V}_r^*\times\mathcal{E}_r^*\subset\mathcal{Z}_a$ be its *neighbourhood* such that on $\mathcal{Z}_r$, $(\bar{u},\bar{p},\bar{\xi}^*)$ is the only critical point of Ξ. The following extremum results are of fundamental importance in the stability analysis and critical-point theory of nonlinear dynamical systems.

Theorem 4.1 (triality theorem). *Suppose that $(\bar{u},\bar{p},\bar{\xi}^*)\in\mathcal{Z}_c$, and $\mathcal{Z}_r$ is a neighbourhood of $(\bar{u},\bar{p},\bar{\xi}^*)$. If the functional $G_{\xi^*}(u)=\langle\Lambda(u);\bar{\xi}^*\rangle$ is concave on $\mathcal{U}_r$, then on $\mathcal{Z}_r$,*

$$\begin{aligned}\Xi(\bar{u},\bar{p},\bar{\xi}^*)&=\min_u\max_p\min_{\xi^*}\Xi(u,p,\xi^*)=\max_p\min_u\min_{\xi^*}\Xi(u,p,\xi^*)\\&=\min_{u,\xi^*}\max_p\Xi(u,p,\xi^*)=\max_p\min_{u,\xi^*}\Xi(u,p,\xi^*).\end{aligned}\tag{4.1}$$

However, if $G_{\xi^*}(u)$ is convex on $\mathcal{U}_r$, then on $\mathcal{Z}_r$ we have either

$$\begin{aligned}\Xi(\bar{u},\bar{p},\bar{\xi}^*)&=\min_u\max_p\min_{\xi^*}\Xi(u,p,\xi^*)=\min_p\max_u\min_{\xi^*}\Xi(u,p,\xi^*)\\&=\min_{\xi^*,u}\max_p\Xi(u,p,\xi^*)=\min_{p,\xi^*}\max_u\Xi(u,p,\xi^*)\end{aligned}\tag{4.2}$$

or

$$\begin{aligned}\Xi(\bar{u},\bar{p},\bar{\xi}^*)&=\max_u\min_{\xi^*}\max_p\Xi(u,p,\xi^*)=\max_p\min_{\xi^*}\max_u\Xi(u,p,\xi^*)\\&=\min_{\xi^*}\max_{u,p}\Xi(u,p,\xi^*)=\max_{u,p}\min_{\xi^*}\Xi(u,p,\xi^*).\end{aligned}\tag{4.3}$$

Proof. Since $E^*\in\check{\Gamma}(\mathcal{E}_a^*)$, $K^*\in\check{\Gamma}(\mathcal{V}_a^*)$, if $G_{\xi^*}(u)=\langle\Lambda(u);\bar{\xi}^*\rangle$ is concave on $\mathcal{U}_r$, then, for the given $\bar{\xi}^*$, $\Xi\in\check{\Gamma}(\mathcal{U}_r)\times\hat{\Gamma}(\mathcal{V}_a^*)$ is a saddle functional. Thus the equality (4.1) follows from the saddle-Lagrangian duality theorem (cf., for example, Gao

2000*a*). However, if $\langle \Lambda(u); \bar{\xi}^* \rangle$ is convex on $\mathcal{U}_{\rm r}$, then, for any given $\xi^* \in \mathcal{E}_{\rm r}^*$, the extended Lagrangian $\Xi \in \hat{\Gamma}(\mathcal{U}_{\rm r}) \times \hat{\Gamma}(\mathcal{V}_{\rm a}^*)$ is a *super-critical functional* (see Gao 2000*a*). By the *super-Lagrangian duality theorem* proved in Gao (2000*a*), we have either (4.2) or (4.3). ■

Theorem 4.2 (tri-duality theorem). *Suppose that* $(\bar{u}, \bar{p}, \bar{\xi}^*) \in \mathcal{Z}_{\rm c}$ *is a critical point of* Ξ *and* $\mathcal{Z}_{\rm r} = \mathcal{U}_{\rm r} \times \mathcal{V}_{\rm r}^* \times \mathcal{E}_{\rm r}^*$ *is a neighbourhood of* $(\bar{u}, \bar{p}, \bar{\xi}^*)$ *such that* $\mathcal{V}_{\rm r}^* \times \mathcal{E}_{\rm r}^* \subset \mathcal{W}_{\rm s}^*$. *If* $\langle \Lambda(u); \bar{\xi}^* \rangle$ *is concave on* $\mathcal{U}_{\rm r}$, *then*

$$\Pi(\bar{u}) = \min_{u \in \mathcal{U}_{\rm r}} \Pi(u) \quad \textit{if and only if } \Pi^{\rm d}(\bar{p}, \bar{\xi}^*) = \max_{p \in \mathcal{V}_{\rm r}^*} \min_{\xi^* \in \mathcal{E}_{\rm r}^*} \Pi^{\rm d}(p, \xi^*). \tag{4.4}$$

However, if $\langle \Lambda(u); \bar{\xi}^* \rangle$ *is convex on* $\mathcal{U}_{\rm r}$, *then*

$$(\bar{u}) = \min_{u \in \mathcal{U}_{\rm r}} \Pi(u) \quad \textit{if and only if } \Pi^{\rm d}(\bar{p}, \bar{\xi}^*) = \min_{(p,\xi^*) \in \mathcal{W}_{\rm s}^*} \Pi^{\rm d}(p, \xi^*), \tag{4.5}$$

$$\Pi(\bar{u}) = \max_{u \in \mathcal{U}_{\rm r}} \Pi(u) \quad \textit{if and only if } \Pi^{\rm d}(\bar{p}, \bar{\xi}^*) = \max_{p \in \mathcal{V}_{\rm r}^*} \min_{\xi^* \in \mathcal{E}_{\rm r}^*} \Pi^{\rm d}(p, \xi^*). \tag{4.6}$$

Proof. This theorem can be proved by combining Lemma 3.3 and the triality theorem. ■

As we have seen that there are two variational arguments p and ξ^* in the canonical dual action problem, subjected to the constraint $(p, \xi^*) \in \mathcal{W}_{\rm s}^*$. The so-called *polar variational method* proposed by Gao (2000*a*) in static systems can be used to relax this constraint.

5. The bi-polarity principle in non-smooth elastodynamics

The goal of this section is to discuss the duality principles in geometrically linear non-conservative Hamilton systems, i.e. for a given total action $\Pi(u)$, there exists a linear operator $\Lambda : \mathcal{U} \to \mathcal{E}$ and canonical functionals $F \in \Gamma_{\rm G}(\mathcal{U}_{\rm a})$, $E \in \Gamma_{\rm G}(\mathcal{E}_{\rm a})$ such that $\Pi(u) = K(\partial_t u) - E(\Lambda u) + F(u)$. A general framework was established recently by Gao (2001*a*). Here we shall study the particularly important problems and concrete applications in boundary/initial-value problems of non-smooth elastodynamics with dissipation.

Let a deformable body in an undeformed state occupy an open, simply connected, bounded domain $\Omega_t = \Omega \times I \subset \mathbb{R}^3 \times \mathbb{R}^+$ with boundary $\Gamma = \partial\Omega_t = \partial\Omega \times \partial I$. Let the general admissible deformation space $\mathcal{U}$ be a subset of Lebesgue integrable space $\mathcal{L}^\alpha(\Omega_t; \mathbb{R}^3)$ $(1 < \alpha < \infty)$, in which certain differentiable conditions are given.† The dual space $\mathcal{U}^* \subset \mathcal{L}^{\alpha^*}(\Omega_t; \mathbb{R}^3)$ of $\mathcal{U}$ is the admissible force space, α^* is the dual number of α $((1/\alpha) + (1/\alpha^*) = 1)$. The element $\boldsymbol{u}^* \in \mathcal{U}^*$ is specified as the body force in the domain Ω_t, and the surface traction on the boundary Γ. The bilinear form $\langle *, * \rangle : \mathcal{U} \times \mathcal{U}^* \to \mathbb{R}$ is defined by

$$\langle \boldsymbol{u}, \boldsymbol{u}^* \rangle = \int_{\Omega_t} {\rm e}^{\omega(t,\boldsymbol{x})} \boldsymbol{u} \cdot \boldsymbol{u}^* \, {\rm d}\Omega_t + \int_{\Gamma} {\rm e}^{\omega(t,\boldsymbol{x})} \boldsymbol{u} \cdot \boldsymbol{u}^* \, {\rm d}\Gamma,$$

where $\omega(\boldsymbol{x}, t) : \Omega_t \to \mathbb{R}$ is a given weight function.

† Since this paper does not concern the existence of stationary points, these differentiable conditions are not important for our current discussion on extremality conditions of the non-convex variational problems.

In mixed boundary/initial-value problems, the boundary Γ can be split into two disjoint parts, i.e. $\Gamma = \Gamma_{\rm f} \cup \Gamma_{\rm u}$, where $\Gamma_{\rm f} = \partial\Omega_{\rm f} \cup \partial I_{\rm f}$, and $\Gamma_{\rm u} = \partial\Omega_{\rm u} \cup \partial I_{\rm u}$ satisfying $\Gamma_{\rm f} \cap \Gamma_{\rm u} = \emptyset$. On $\Gamma_{\rm f}$, the surface traction $\bar{\boldsymbol{t}}$ is given; while on the remaining part $\Gamma_{\rm u}$, the displacement is prescribed. For problems with homogeneous boundary condition, the admissible deformation space $\mathcal{U}_{\rm a} \subset \mathcal{U}$ can be defined by

$$\mathcal{U}_{\rm a} = \{\boldsymbol{u} \in \mathcal{U} \mid \boldsymbol{u}(\boldsymbol{x}, t) = 0 \ \forall (\boldsymbol{x}, t) \in \Gamma_{\rm u}\}. \tag{5.1}$$

If the external force is given as $\bar{\boldsymbol{u}}^*(\boldsymbol{x}, t) = \{\bar{\boldsymbol{b}}(\boldsymbol{x}, t)(\text{in } \Omega_t);\ \bar{\boldsymbol{t}}(\boldsymbol{x}, t)(\text{in } \Gamma_{\rm f})\}$, the external potential $F : \mathcal{U} \to \bar{\mathbb{R}}$ can be defined as

$$F(\boldsymbol{u}) = \langle \boldsymbol{u}, \bar{\boldsymbol{u}}^* \rangle = \int_{\Omega_t} \mathrm{e}^{\omega(\boldsymbol{x},t)} \boldsymbol{u} \cdot \bar{\boldsymbol{b}} \, \mathrm{d}\Omega_t + \int_{\Gamma_{\rm f}} \mathrm{e}^{\omega(\boldsymbol{x},t)} \boldsymbol{u} \cdot \bar{\boldsymbol{t}} \, \mathrm{d}\Gamma, \tag{5.2}$$

which is a linear functional defined on a convex domain $\mathcal{U}_{\rm a}$. Thus, the duality relation

$$\boldsymbol{u}^* = DF(\boldsymbol{u}) = \begin{cases} \bar{\boldsymbol{b}} & \text{in } \Omega_t, \\ \bar{\boldsymbol{t}} & \text{on } \Gamma_{\rm f}, \end{cases} \qquad \forall \boldsymbol{u} \in \mathcal{U}_{\rm a} \tag{5.3}$$

gives the natural boundary condition.

In Newtonian systems, the kinetic energy is a quadratic functional on the Hilbert space $\mathcal{V}_{\rm a} = \mathcal{V} = \mathcal{H}(\Omega_t; \mathbb{R}^3)$:

$$K(\boldsymbol{v}) = \tfrac{1}{2} \langle \boldsymbol{v}, \rho \boldsymbol{v} \rangle = \int_{\Omega_t} \tfrac{1}{2} \mathrm{e}^{\omega(\boldsymbol{x},t)} \rho \boldsymbol{v} \cdot \boldsymbol{v} \, \mathrm{d}\Omega_t.$$

For the linear operator $\partial_t = \partial/\partial t$, its adjoint $\partial_t^* : \mathcal{V}^* \to \mathcal{U}^*$ can be obtained by using integration by parts:

$$\begin{aligned}\langle \partial_t \boldsymbol{u}, \boldsymbol{p} \rangle &= \int_{\Omega_t} \mathrm{e}^{\omega(\boldsymbol{x},t)} \frac{\partial \boldsymbol{u}}{\partial t} \cdot \boldsymbol{p} \, \mathrm{d}\Omega_t = \int_{\Omega_t} [\mathrm{e}^{\omega(\boldsymbol{x},t)} \boldsymbol{u} \cdot \boldsymbol{p}]_{,t} \, \mathrm{d}\Omega_t - \int_{\Omega_t} \boldsymbol{u} \cdot (\mathrm{e}^{\omega(\boldsymbol{x},t)} \boldsymbol{p})_{,t} \, \mathrm{d}\Omega_t \\ &= \int_{\Omega} \mathrm{e}^{\omega(\boldsymbol{x},t)} n \boldsymbol{u} \cdot \boldsymbol{p}|_{\partial I_{\rm f}} \, \mathrm{d}\Omega - \int_{\Omega_t} \mathrm{e}^{\omega(\boldsymbol{x},t)} \boldsymbol{u} \cdot (\boldsymbol{p}_{,t} + \omega_{,t} \boldsymbol{p}) \, \mathrm{d}\Omega_t = \langle \boldsymbol{u}, \partial_t^* \boldsymbol{p} \rangle.\end{aligned}$$

Thus, the adjoint operator ∂_t^* is defined by

$$\partial_t^* = \begin{cases} -\dfrac{\partial}{\partial t} - \omega_{,t} & \text{in } \Omega_t, \\ n & \text{on } \partial I_{\rm f}, \end{cases}$$

where $n = \pm 1$ is a unit vector normal to the time boundary ∂I. For linear time-dissipative systems, the weight function is usually a linear function of $t \in I$, say $\omega = \nu t$, where $\nu > 0$ is a given constant (see Gao 2000*a*).

For an infinitesimal deformation system, the canonical strain measure $\xi = \boldsymbol{\epsilon}$ is a symmetrical tensor field

$$\boldsymbol{\epsilon} = \Lambda \boldsymbol{u} = \tfrac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\rm T}). \tag{5.4}$$

Thus, the geometrical operator $\Lambda = \nabla_{\rm s} = \frac{1}{2}(\nabla + \nabla^{\rm T})$ is a linear differential mapping, which assigns to each displacement vector $\boldsymbol{u} \in \mathcal{U}$ a symmetrical tensor in $\mathbb{R}^{3\times 3}(\Omega_t)$. Let

$$\mathcal{E} = \{\boldsymbol{\epsilon} \in \mathcal{L}^{\beta}(\Omega_t; \mathbb{R}^{3\times 3}) \mid \boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\rm T} \text{ in } \Omega_t\}$$

be a space of all second-order, symmetrical, Lebesgue integrable tensor functions with domain in Ω_t and range in $\mathbb{R}^{3\times 3}$. For each given $\boldsymbol{\epsilon} \in \mathcal{E}$, the conjugate stress measure $\boldsymbol{\sigma} = \xi^*$ is also a second-order symmetrical tensor field, defined by the general non-smooth constitutive law

$$\boldsymbol{\sigma} \in \partial U(\boldsymbol{\epsilon}) : \mathcal{E} \to \mathcal{E}^*, \tag{5.5}$$

where $U : \mathcal{E} \to \bar{\mathbb{R}}$ is the so-called stored strain energy density, and

$$\mathcal{E}^* = \{\boldsymbol{\sigma} \in \mathcal{L}^{\beta^*}(\Omega_t; \mathbb{R}^{3\times 3}) \mid \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}} \text{ in } \Omega_t\}$$

is the dual space of $\mathcal{E}$ with $(1/\beta) + (1/\beta^*) = 1$. The bilinear form between $\mathcal{E}$ and $\mathcal{E}^*$ can be defined by

$$\langle \boldsymbol{\epsilon}; \boldsymbol{\sigma} \rangle = \int_{\Omega_t} \mathrm{e}^{\omega(t,\boldsymbol{x})} \boldsymbol{\epsilon} : \boldsymbol{\sigma} \, \mathrm{d}\Omega_t,$$

where $\boldsymbol{\epsilon} : \boldsymbol{\sigma} = \mathrm{tr}(\boldsymbol{\epsilon} \cdot \boldsymbol{\sigma})$. By using the Gauss–Green formula, we have

$$\begin{aligned}
\langle \Lambda \boldsymbol{u}; \boldsymbol{\sigma} \rangle &= \int_{\Omega_t} \mathrm{e}^{\omega(t,\boldsymbol{x})} (\nabla_{\mathrm{s}} \boldsymbol{u}) : \boldsymbol{\sigma} \, \mathrm{d}\Omega_t \\
&= \int_{\Omega} [\nabla \cdot (\mathrm{e}^{\omega(t,\boldsymbol{x})} \boldsymbol{u} \cdot \boldsymbol{\sigma}) - \boldsymbol{u} \cdot (\nabla \cdot (\mathrm{e}^{\omega(t,\boldsymbol{x})} \boldsymbol{\sigma}))] \, \mathrm{d}\Omega_t \\
&= \int_I \oint_{\partial\Omega} \mathrm{e}^{\omega(\boldsymbol{x},t)} \boldsymbol{u} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \mathrm{d}\partial\Omega \, \mathrm{d}t - \int_{\Omega_t} \mathrm{e}^{\omega(\boldsymbol{x},t)} \boldsymbol{u} \cdot [\nabla \cdot \boldsymbol{\sigma} + (\nabla\omega(\boldsymbol{x},t)) \cdot \boldsymbol{\sigma}] \, \mathrm{d}\Omega_t \\
&= \langle \boldsymbol{u}, \Lambda^* \boldsymbol{\sigma} \rangle,
\end{aligned}$$

Author: rearrangement onto 4 lines OK?

where $\boldsymbol{n} \in \mathbb{R}^3$ is a unit vector normal to $\partial\Omega$. Thus, the adjoint operator $\Lambda^* : \mathcal{E}^* \to \mathcal{U}^*$ associated with these bilinear forms is defined by

$$\Lambda^* \boldsymbol{\sigma} = \begin{cases} -\nabla \cdot \boldsymbol{\sigma} - (\nabla\omega(\boldsymbol{x},t)) \cdot \boldsymbol{\sigma} & \text{in } \Omega_t, \\ \boldsymbol{n} \cdot \boldsymbol{\sigma} & \text{on } \partial\Omega. \end{cases}$$

Let the admissible strain space $\mathcal{E}_{\mathrm{a}}$ be a subset of $\mathcal{E}$, on which, the stored strain energy density $U(\boldsymbol{\epsilon})$ is finite and convex; let $\mathcal{E}_{\mathrm{a}}^* \subset \mathcal{E}^*$ be the range of the constitutive mapping $\partial U : \mathcal{E}_{\mathrm{a}} \to \mathcal{E}^*$ such that the Fenchel duality relations

$$\boldsymbol{\sigma} \in \partial U(\boldsymbol{\epsilon}) \quad \Leftrightarrow \quad \boldsymbol{\epsilon} \in \partial U^*(\boldsymbol{\sigma}) \quad \Leftrightarrow \quad U(\boldsymbol{\epsilon}) + U^*(\boldsymbol{\sigma}) = \mathrm{tr}(\boldsymbol{\epsilon} \cdot \boldsymbol{\sigma})$$

hold on $\mathcal{E}_{\mathrm{a}} \times \mathcal{E}_{\mathrm{a}}^*$. Thus, on the kinematically admissible space $\mathcal{U}_k = \{\boldsymbol{u} \in \mathcal{U}_{\mathrm{a}} \mid \partial_t \boldsymbol{u} \in \mathcal{V}_{\mathrm{a}},\ \nabla_{\mathrm{s}} \boldsymbol{u} \in \mathcal{E}_{\mathrm{a}}\}$, the primal problem

$$\begin{aligned}
\Pi(\boldsymbol{u}) &= K(\partial_t \boldsymbol{u}) - E(\Lambda \boldsymbol{u}) + F(\boldsymbol{u}) \\
&= \int_{\Omega_t} \mathrm{e}^{\omega(t,\boldsymbol{x})} (\tfrac{1}{2} \rho \boldsymbol{u}_{,t} \cdot \boldsymbol{u}_{,t} - U(\nabla_{\mathrm{s}} \boldsymbol{u})) \, \mathrm{d}\Omega_t + F(\boldsymbol{u}) \to \mathrm{sta} \quad \forall \boldsymbol{u} \in \mathcal{U}_k
\end{aligned}$$

is a geometrically linear variational problem. The criticality condition of Π leads to the standard geometrically linear trio-canonical forms in the domain Ω_t,

$$\left.\begin{aligned}
\boldsymbol{v} = \partial_t \boldsymbol{u} = \boldsymbol{u}_{,t}, \qquad \boldsymbol{\epsilon} = \Lambda \boldsymbol{u} = \nabla_{\mathrm{s}} \boldsymbol{u}, \\
\boldsymbol{p} = DK(\boldsymbol{v}) = \rho \boldsymbol{v}, \qquad \boldsymbol{\sigma} \in \partial U(\boldsymbol{\epsilon}), \qquad \boldsymbol{u}^* \in \partial F(\boldsymbol{u}), \\
\boldsymbol{u}^* = -\partial_t^* \boldsymbol{p} + \Lambda^* \boldsymbol{\sigma} = \boldsymbol{p}_{,t} + \omega_{,t} \boldsymbol{p} - \nabla \cdot \boldsymbol{\sigma} - (\nabla\omega) \cdot \boldsymbol{\sigma},
\end{aligned}\right\} \tag{5.6}$$

and the natural boundary condition $\boldsymbol{u}^* = DF(\boldsymbol{u}) = \bar{\boldsymbol{t}}$ on Γ_{f}.

To see a unified beauty of duality theory in geometrically linear systems, we introduce two product spaces $\mathcal{W} = \mathcal{V} \times \mathcal{E}$ and $\mathcal{W}^* = \mathcal{V}^* \times \mathcal{E}^*$, placed in duality by the bilinear form $\langle * : * \rangle : \mathcal{W} \times \mathcal{W}^* \to \mathbb{R}$. For each given $\boldsymbol{\varpi} = (\boldsymbol{v}, -\boldsymbol{\epsilon})^{\mathrm{T}} \in \mathcal{W}$, the canonical function $W : \mathcal{W} \to \bar{\mathbb{R}}$, defined by

$$W(\boldsymbol{\varpi}) = W(\boldsymbol{v}, \boldsymbol{\epsilon}) = K(\boldsymbol{v}) - E(\boldsymbol{\epsilon}) = \int_{\Omega_t} \mathrm{e}^{\omega(\boldsymbol{x},t)} [\tfrac{1}{2}\rho \boldsymbol{v} \cdot \boldsymbol{v} - U(\boldsymbol{\epsilon})] \, \mathrm{d}\Omega_t, \tag{5.7}$$

is called the *stored action* of the system. For each given $\boldsymbol{\varpi}^* = (\boldsymbol{p}, \boldsymbol{\sigma})^{\mathrm{T}} \in \mathcal{W}^*$, the conjugate functional of W is simply defined by

$$\begin{aligned} W^*(\boldsymbol{\varpi}^*) &= \mathrm{ext}\{\langle \boldsymbol{\varpi} : \boldsymbol{\varpi}^* \rangle - W(\boldsymbol{\varpi}) \mid \forall \boldsymbol{\varpi} \in \mathcal{W}\} \\ &= \mathrm{ext}\{\langle \boldsymbol{v}, \boldsymbol{p} \rangle - K(\boldsymbol{v}) - \langle \boldsymbol{\epsilon}; \boldsymbol{\epsilon}^* \rangle + E(\boldsymbol{\epsilon}) \mid \forall \boldsymbol{v} \in \mathcal{V},\ \boldsymbol{\epsilon} \in \mathcal{E}\} \\ &= \int_{\Omega_t} \mathrm{e}^{\omega(\boldsymbol{x},t)} [\tfrac{1}{2}\rho^{-1} \boldsymbol{p} \cdot \boldsymbol{p} - U^*(\boldsymbol{\sigma})] \, \mathrm{d}\Omega_t = K^*(\boldsymbol{p}) - E^*(\boldsymbol{\epsilon}^*). \end{aligned}$$

Thus, the Fenchel duality relations

$$\boldsymbol{\varpi}^* \in \partial W(\boldsymbol{\varpi}) \quad \Leftrightarrow \quad \boldsymbol{\varpi} \in \partial W^*(\boldsymbol{\varpi}^*) \quad \Leftrightarrow \quad W(\boldsymbol{\varpi}) + W^*(\boldsymbol{\varpi}) = \langle \boldsymbol{\varpi} : \boldsymbol{\varpi}^* \rangle$$

holds on $\mathcal{W} \times \mathcal{W}^*$.

Let $\mathcal{W}_{\mathrm{a}} = \mathcal{V}_{\mathrm{a}} \times \mathcal{E}_{\mathrm{a}}$. By introducing a space-time operator $\Upsilon = (\partial_t, -\Lambda)^{\mathrm{T}} : \mathcal{U}_{\mathrm{a}} \to \mathcal{W}_{\mathrm{a}}$, the geometrical equations in (5.6) can be written in the vector form

$$\boldsymbol{\varpi}(\boldsymbol{u}) = \begin{pmatrix} \boldsymbol{v}(\boldsymbol{u}) \\ -\boldsymbol{\epsilon}(\boldsymbol{u}) \end{pmatrix} = \Upsilon \boldsymbol{u} = \begin{pmatrix} \partial_t \boldsymbol{u} \\ -\Lambda \boldsymbol{u} \end{pmatrix} \in \mathcal{W}_{\mathrm{a}}, \tag{5.8}$$

and, for any given $\boldsymbol{u} \in \mathcal{U}_{\mathrm{a}}$, the balance operator $\Upsilon^* : \mathcal{W}^*_{\mathrm{a}} = \mathcal{V}^*_{\mathrm{a}} \times \mathcal{E}^*_{\mathrm{a}} \to \mathcal{U}^*$ can be defined by

$$\langle \boldsymbol{u}, \Upsilon^* \varpi^* \rangle = \langle \Upsilon \boldsymbol{u} : \boldsymbol{\varpi}^* \rangle \Rightarrow \Upsilon^* = (\partial_t^*, -\Lambda^*). \tag{5.9}$$

Thus, the Lagrangian $\Xi : \mathcal{Z} = \mathcal{U} \times \mathcal{W}^* \to \bar{\mathbb{R}}$ in geometrically linear dynamical systems has the standard form

$$\begin{aligned} \Xi(\boldsymbol{u}, \boldsymbol{\varpi}^*) &= \langle \Upsilon \boldsymbol{u} : \boldsymbol{\varpi}^* \rangle - W^*(\boldsymbol{\varpi}^*) + F(\boldsymbol{u}) \\ &= \int_{\Omega_t} \mathrm{e}^{\omega(\boldsymbol{x},t)} [\boldsymbol{u}_{,t} \cdot \boldsymbol{p} - (\nabla_{\mathrm{s}} \boldsymbol{u}) : \boldsymbol{\sigma} - \tfrac{1}{2}\rho^{-1} \boldsymbol{p} \cdot \boldsymbol{p} + U^*(\boldsymbol{\sigma})] \, \mathrm{d}\Omega_t + F(\boldsymbol{u}), \end{aligned}$$

which is a canonical functional on $\mathcal{U}_{\mathrm{a}} \times \mathcal{Z}_{\mathrm{a}}$. For any given $\boldsymbol{\varpi}^* \in \mathcal{W}^*_{\mathrm{a}}$, the dual action can be obtained by the standard canonical Lagrange dual transformation

$$\Pi^{\mathrm{d}}(\boldsymbol{\varpi}^*) = \mathrm{sta}\{\Xi(\boldsymbol{u}, \boldsymbol{\varpi}^*) \mid \forall \boldsymbol{\varpi}^* \in \mathcal{W}^*_{\mathrm{a}}\} = F^*(\Upsilon^* \boldsymbol{\varpi}^*) - W^*(\boldsymbol{\varpi}^*), \tag{5.10}$$

where $F^*(\boldsymbol{u}^*)$ is defined by

$$F^*(\Upsilon^* \boldsymbol{\varpi}^*) = \mathrm{sta}\{\langle \Upsilon \boldsymbol{u} : \boldsymbol{\varpi}^* \rangle + F(\boldsymbol{u}) \mid \forall \boldsymbol{u} \in \mathcal{U}_{\mathrm{a}}\}. \tag{5.11}$$

The criticality condition of this problem leads to the balance equation

$$0 \in \Upsilon^* \boldsymbol{\varpi}^* + \partial F(\bar{\boldsymbol{u}}). \tag{5.12}$$

For each critical point $\bar{\boldsymbol{u}} \in \mathcal{U}_{\rm a}$, the general solution for this linear, non-homogeneous equation can be written as

$$\boldsymbol{\varpi}^* = \boldsymbol{\varpi}^*_o + \boldsymbol{\varpi}^*_{\rm p}, \qquad \Upsilon^* \boldsymbol{\varpi}^*_o = 0, \qquad 0 \in \Upsilon^* \boldsymbol{\varpi}^*_{\rm p} + \partial F(\bar{\boldsymbol{u}}),$$

where $\boldsymbol{\varpi}^*_{\rm p} \in \mathcal{W}^*_{\rm a}$ is a particular solution and $\underline{\boldsymbol{\varpi}^*_o}$ is a homogeneous solution. Author: what is subscript ‘o’ here?

For a mixed boundary-value problem such that $F(\boldsymbol{u}) = \langle \boldsymbol{u}, \bar{\boldsymbol{u}}^* \rangle$ is a linear functional, the dual feasible space is a hyper-plane in the product space $\mathcal{W}^*_{\rm a}$:

$$\mathcal{W}^*_{\rm s} = \{\boldsymbol{\varpi}^* \in \mathcal{W}^*_{\rm a} \mid \langle \boldsymbol{u}, \Upsilon^* \boldsymbol{\varpi}^* + \bar{\boldsymbol{u}}^* \rangle = 0 \ \forall \boldsymbol{u} \in \mathcal{U}_{\rm a}\}.$$

Thus, on $\mathcal{W}^*_{\rm s}$, we have $F^*(\Upsilon^* \varpi^*) = 0$, and the dual variational problem takes a very simple form:

$$\Pi^{\rm d}(\boldsymbol{\varpi}^*) = -W^*(\boldsymbol{\varpi}^*) = \int_{\Omega_t} {\rm e}^{\omega(\boldsymbol{x},t)} [U^*(\boldsymbol{\sigma}) - \tfrac{1}{2}\rho^{-1} \boldsymbol{p} \cdot \boldsymbol{p}] \, {\rm d}\Omega_t \to {\rm sta} \quad \forall \boldsymbol{\varpi}^* \in \mathcal{W}^*_{\rm s}.$$

In order to relax the balance constraint in $\mathcal{W}^*_{\rm s}$, we let $\boldsymbol{\varpi}^* = \boldsymbol{\varpi}^*_o + \boldsymbol{\varpi}^*_{\rm p}$. For the given source $\bar{\boldsymbol{u}}^*$, the particular solution $\boldsymbol{\varpi}^*_{\rm p}$ can by determined by solving the linear equation $\Upsilon^* \boldsymbol{\varpi}^*_{\rm p} + \bar{\boldsymbol{u}}^* = 0$. Let $\mathcal{U}^o$ be the so-called *polar configuration space* (see Gao 2000*a*), placed in duality with $\mathcal{U}^{o*}$ by the bilinear form $\langle *, * \rangle : \mathcal{U}^o \times \mathcal{U}^{o*} \to \mathbb{R}$. For the given geometrically linear system and $\Upsilon : \mathcal{U} \to \mathcal{W}$, a linear mapping $\Upsilon^o : \mathcal{U}^o \to \mathcal{W}^*$ is called a *null-source polar operator* of Υ if

$$\langle \Upsilon \boldsymbol{u} : \Upsilon^o \boldsymbol{u}^o \rangle = 0. \tag{5.13}$$

Clearly, if there exists a subset $\mathcal{U}^o_{\rm a} \subset \mathcal{U}^o$ such that (5.13) holds for any given $\boldsymbol{u}^o \in \mathcal{U}^o_{\rm a} \subset \mathcal{U}^o$, then

$$\Upsilon^{o*} \Upsilon \boldsymbol{u} = 0 \in \mathcal{U}^{o*}, \tag{5.14}$$

which is the so-called *compatibility condition*. On the other hand, if (5.13) holds for any $\boldsymbol{u} \in \mathcal{U}_{\rm a}$, then the equation

$$\Upsilon^* \Upsilon^o \boldsymbol{u}^o = 0 \in \mathcal{U}^* \tag{5.15}$$

is called the *polar compatibility condition* (Gao 2000*a*). For a given operator $\Upsilon : \mathcal{U}_{\rm a} \to \mathcal{W}$, the *null-source admissible polar configuration space* $\mathcal{U}^o_{\rm a} \subset \mathcal{U}^o$ is defined by

$$\mathcal{U}^o_{\rm a} = \{\boldsymbol{u}^o \in \mathcal{U}^o \mid \langle \Upsilon \boldsymbol{u} : \Upsilon^o \boldsymbol{u}^o \rangle = 0 \ \forall \boldsymbol{u} \in \mathcal{U}_{\rm a}\}. \tag{5.16}$$

Thus, for any given $\boldsymbol{u}^o \in \mathcal{U}^o_{\rm a}$, the balance condition $\boldsymbol{\varpi}^* = \boldsymbol{\varpi}^*_o + \boldsymbol{\varpi}^*_{\rm p} \in \mathcal{W}^*_{\rm s}$ is satisfied and $\boldsymbol{\varpi}^*_o = \Upsilon^o \boldsymbol{u}^o$ is in the null space of the balance operator Υ^*. A diagrammatic representation for the geometrically linear polar system is shown in figure 2.

There are many choices for the null-source polar operator. In static systems, the polar configuration variable $\boldsymbol{u}^o$ possesses certain beautiful geometrical meanings (see Gao 2000*a*). For dynamical systems such that the geometrical operator $\Upsilon = (\partial_t, \Lambda)^{\rm T} : \mathcal{U} \to \mathcal{W} = \mathcal{V} \times \mathcal{E}$ is linear, if $\partial_t \Lambda = \Lambda \partial_t$, we can simply let $\Upsilon^o = (\Lambda^*, \partial^*_t)^{\rm T}$ (see Gao 2001*a*). Thus,

$$\langle \Upsilon \boldsymbol{u} : \Upsilon^* \boldsymbol{u}^o \rangle = \langle \partial_t \boldsymbol{u}, \Lambda^* \boldsymbol{u}^o \rangle - \langle \Lambda \boldsymbol{u}; \partial^*_t \boldsymbol{u}^o \rangle = \langle \Lambda \partial_t \boldsymbol{u} - \partial_t \Lambda \boldsymbol{u}, \boldsymbol{u}^o \rangle = 0.$$

Let $\boldsymbol{\varpi}^*_{\rm p} \in \mathcal{W}^*_{\rm s}$ be a particular solution of the linear balance equation $\Upsilon^* \boldsymbol{\varpi}^* + \bar{\boldsymbol{u}}^* = 0$. Replacing $\boldsymbol{\varpi}^*$ by $\boldsymbol{\varpi}^* = \Upsilon^o \boldsymbol{u}^o + \boldsymbol{\varpi}^*_{\rm p} \in \mathcal{W}^*_{\rm s}$, and letting $W^o(\boldsymbol{u}^o) = W^*(\Upsilon^o \boldsymbol{u}^o + \boldsymbol{\varpi}^*_{\rm p})$, the canonical Hamiltonian Θ can be written as

$$\Theta^o(\boldsymbol{u}, \boldsymbol{u}^o) = W^o(\Upsilon^o \boldsymbol{u}^o) + \langle \boldsymbol{u}, \bar{\boldsymbol{u}}^* \rangle : \mathcal{U} \times \mathcal{U}^o \to \bar{\mathbb{R}}, \tag{5.17}$$

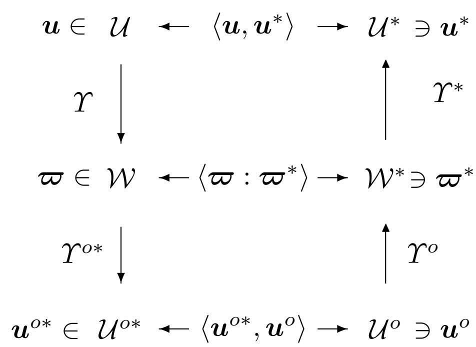

Figure 2. Structure of geometrically linear system and its polar.

which we call the *polar Hamiltonian*. The associated *polar Lagrangian* $\Xi^o : \mathcal{U} \times \mathcal{U}^o \to \bar{\mathbb{R}}$ is defined by

$$\Xi^o(\boldsymbol{u}, \boldsymbol{u}^o) = \langle \Upsilon \boldsymbol{u} : \Upsilon^o \boldsymbol{u}^o \rangle - W^o(\Upsilon^o \boldsymbol{u}^o). \tag{5.18}$$

Clearly, $\Xi^o \in \Gamma_0(\mathcal{U}) \times \Gamma_0(\mathcal{U}^o)$ is an extended canonical functional. On $\mathcal{U}_\mathrm{a} \times \mathcal{U}^o_\mathrm{a}$, the criticality condition leads to the polar governing equations

$$D\Xi^o(\boldsymbol{u}, \boldsymbol{u}^o) = 0 \quad \Leftrightarrow \quad \begin{cases} \Upsilon^{o*} DW^o(\Upsilon^o \boldsymbol{u}^o) = \Upsilon^{o*} \Upsilon \boldsymbol{u}, \\ \Upsilon^* \Upsilon^o \boldsymbol{u}^o = 0. \end{cases} \tag{5.19}$$

By the fact that $W \in \Gamma_\mathrm{G}(\mathcal{W}_\mathrm{a})$, we have $(W^*(\boldsymbol{\varpi}^*))^* = W(\boldsymbol{\varpi})$. Thus, for any given $\boldsymbol{u} \in \mathcal{U}_k$,

$$\Pi(\boldsymbol{u}) = \mathrm{ext}\{\Xi^o(\boldsymbol{u}, \boldsymbol{u}^o) \mid \forall \boldsymbol{u}^o \in \mathcal{U}^o\}.$$

On the other hand, by introducing the *polar feasible space*

$$\mathcal{U}^o_k = \{\boldsymbol{u}^o \in \mathcal{U}^o_\mathrm{a} \mid \Upsilon^o \boldsymbol{u}^o \in \mathcal{W}^*_\mathrm{a}\}, \tag{5.20}$$

the *polar action* can be obtained by

$$\Pi^o(\boldsymbol{u}^o) = \mathrm{sta}\{\Xi^o(\boldsymbol{u}, \boldsymbol{u}^o) \mid \forall \boldsymbol{u} \in \mathcal{U}\} = -W^o(\Upsilon^o \boldsymbol{u}^o). \tag{5.21}$$

Then the *polar variational problem* $(\mathcal{P}^o)$ can be proposed as

$$(\mathcal{P}^o) : \Pi^o(\boldsymbol{u}^o) \to \mathrm{sta} \quad \forall \boldsymbol{u}^o \in \mathcal{U}^o_k. \tag{5.22}$$

The critical condition $D\Pi^o(\boldsymbol{u}^o) = 0$ leads to the polar governing equation:

$$\Upsilon^{o*} DW^o(\Upsilon^o \boldsymbol{u}^o) = \Upsilon^{o*} \Upsilon \boldsymbol{u}, \tag{5.23}$$

where $\boldsymbol{u} \in \mathcal{U}_\mathrm{a}$ is the Lagrange multiplier for the equilibrium condition in $\mathcal{U}^o_\mathrm{a}$.

Theorem 5.1 (bi-polarity theorem). *Suppose that for a given total action* $\Pi(\boldsymbol{u}) = K(\partial_t \boldsymbol{u}) - E(\Lambda \boldsymbol{u}) + \langle \boldsymbol{u}, \bar{\boldsymbol{u}}^* \rangle$ *such that* $K \in \check{\Gamma}(\mathcal{V}_\mathrm{a})$ *and* $E \in \check{\Gamma}(\mathcal{E}_\mathrm{a})$ *are convex, and* $\Pi^o : {\mathcal{U}^o}_k \to \mathbb{R}$ *is the associated polar action. If* $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}}^o)$ *is a critical point of* Ξ^o, *then* $\Pi(\bar{\boldsymbol{u}}) = \Xi^o(\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}}^o) = \Pi^o(\bar{\boldsymbol{u}}^o)$ *and*

$$\Pi(\bar{\boldsymbol{u}}) = \inf_{\boldsymbol{u} \in \mathcal{U}_k} \Pi(\boldsymbol{u}) \quad \Leftrightarrow \quad \inf_{\boldsymbol{u}^o \in \mathcal{U}^o_k} \Pi^o(\boldsymbol{u}^o) = \Pi^o(\bar{\boldsymbol{u}}^o), \tag{5.24}$$

$$\Pi(\bar{\boldsymbol{u}}) = \sup_{\boldsymbol{u} \in \mathcal{U}_k} \Pi(\boldsymbol{u}) \quad \Leftrightarrow \quad \sup_{\boldsymbol{u}^o \in \mathcal{U}^o_k} \Pi^o(\boldsymbol{u}^o) = \Pi^o(\bar{\boldsymbol{u}}^o). \tag{5.25}$$

Proof. For linear operator $\Upsilon = (\partial_t, \Lambda)^{\mathrm{T}} : \mathcal{U} \to \mathcal{W}$, the extended Lagrangian $\Xi : \mathcal{U}_{\mathrm{a}} \times \mathcal{W}_{\mathrm{a}} \to \mathbb{R}$ associated with Π is

$$\Xi(\boldsymbol{u}, \boldsymbol{p}, \boldsymbol{\xi}^*) = \langle \partial_t \boldsymbol{u}, \boldsymbol{p} \rangle - K^*(\boldsymbol{p}) - \langle \Lambda \boldsymbol{u}; \boldsymbol{\xi}^* \rangle + E^*(\boldsymbol{\xi}^*) + \langle \boldsymbol{u}, \bar{\boldsymbol{u}}^* \rangle. \tag{5.26}$$

By lemma 3.3, we know that if $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}}^o, \bar{\boldsymbol{\epsilon}}^*)$ is a critical point of Ξ, then $\Pi(\bar{\boldsymbol{u}}) = \Pi^{\mathrm{d}}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{\epsilon}}^*)$. Replacing $\bar{\boldsymbol{\varpi}}^* = (\bar{\boldsymbol{p}}, \bar{\boldsymbol{\epsilon}}^*)$ by $\Upsilon^o \bar{\boldsymbol{u}}^o + \boldsymbol{\varpi}^*_{\mathrm{p}}$ leads to $\Pi(\bar{\boldsymbol{u}}) = \Xi^o(\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}}^o) = \Pi^o(\bar{\boldsymbol{u}}^o)$.

Since $E^*(\boldsymbol{\xi}^*)$ and $K^*(\boldsymbol{p})$ are convex, for any given $\boldsymbol{u} \in \mathcal{U}_{\mathrm{a}}$, $\Xi \in \hat{\Gamma}(\mathcal{V}^*_{\mathrm{a}}) \times \check{\Gamma}_{\mathrm{G}}(\mathcal{E}^*_{\mathrm{a}})$ is a left-saddle functional, hence

$$\sup_{\boldsymbol{p} \in \mathcal{V}^*_{\mathrm{a}}} \inf_{\boldsymbol{\xi}^* \in \mathcal{E}^*_{\mathrm{a}}} \Xi(\boldsymbol{u}, \boldsymbol{p}, \boldsymbol{\xi}^*) = \inf_{\boldsymbol{\xi}^* \in \mathcal{E}^*_{\mathrm{a}}} \sup_{\boldsymbol{p} \in \mathcal{V}^*_{\mathrm{a}}} \Xi(\boldsymbol{u}, \boldsymbol{p}, \boldsymbol{\xi}^*) = \Pi(\boldsymbol{u}) \quad \forall \boldsymbol{u} \in \mathcal{U}_k.$$

On the other hand, for a given $(\boldsymbol{u}, \boldsymbol{p}) \in \mathcal{U}_{\mathrm{a}} \times \mathcal{V}^*_{\mathrm{a}}$, $\inf_{\boldsymbol{\xi}^* \in \mathcal{E}^*} \Xi(\boldsymbol{u}, \boldsymbol{p}, \boldsymbol{\xi}^*)$ leads to the classical Lagrangian

$$L(\boldsymbol{u}, \boldsymbol{p}) = \inf_{\boldsymbol{\xi}^* \in \mathcal{E}^*} \Xi(\boldsymbol{u}, \boldsymbol{p}, \boldsymbol{\xi}^*) = \langle \partial_t \boldsymbol{u}, \boldsymbol{p} \rangle - K^*(\boldsymbol{p}) - E(\Lambda \boldsymbol{u}) + \langle \boldsymbol{u}, \bar{\boldsymbol{u}}^* \rangle,$$

which is a super-critical functional on $\mathcal{U}_{\mathrm{a}} \times \mathcal{V}^*_{\mathrm{a}}$. By the super-Lagrange duality theory (cf. Gao 2000*a*), we know that

$$\inf_{\boldsymbol{u}} \sup_{\boldsymbol{p}} L(\boldsymbol{u}, \boldsymbol{p}) = \inf_{\boldsymbol{p}} \sup_{\boldsymbol{u}} L(\boldsymbol{u}, \boldsymbol{p}) \quad \text{and} \quad \sup_{\boldsymbol{u}} \sup_{\boldsymbol{p}} L(\boldsymbol{u}, \boldsymbol{p}) = \sup_{\boldsymbol{p}} \sup_{\boldsymbol{u}} L(\boldsymbol{u}, \boldsymbol{p}).$$

Thus,

$$\begin{aligned} \inf_{\boldsymbol{u}} \Pi(\boldsymbol{u}) &= \inf_{\boldsymbol{u}} \sup_{\boldsymbol{p}} \inf_{\boldsymbol{\xi}^*} \Xi(\boldsymbol{u}, \boldsymbol{p}, \boldsymbol{\xi}^*) = \inf_{\boldsymbol{p}} \sup_{\boldsymbol{u}} \inf_{\boldsymbol{\xi}^*} \Xi(\boldsymbol{u}, \boldsymbol{p}, \boldsymbol{\xi}^*) \\ &= \inf_{\boldsymbol{p}} \inf_{\boldsymbol{\xi}^*} \sup_{\boldsymbol{u}} \Xi(\boldsymbol{u}, \boldsymbol{p}, \boldsymbol{\xi}^*) = \inf_{\boldsymbol{p}, \boldsymbol{\xi}^*} \Pi^{\mathrm{d}}(\boldsymbol{p}, \boldsymbol{\xi}^*). \end{aligned}$$

Replacing $(\boldsymbol{p}, \boldsymbol{\xi}^*)$ by $\boldsymbol{\varpi}^* = \Upsilon^o \boldsymbol{u}^o + \boldsymbol{\varpi}^*_{\mathrm{p}}$, we have $\sup_{\boldsymbol{u}} \Xi(\boldsymbol{u}, \boldsymbol{p}(\boldsymbol{u}^o), \boldsymbol{\xi}^*(\boldsymbol{u}^o)) = \Pi^o(\boldsymbol{u}^o)$, and this proves statement (5.24).

Similarly, statement (5.25) can be proved by the bi-duality theorem. ■

For conservative systems with standard bilinear forms ($\omega = 0$), the geometrically linear operator Υ and its polar operator can be chosen as

$$\Upsilon = \begin{pmatrix} \partial/\partial t \\ -\nabla_{\mathrm{s}} \end{pmatrix}, \qquad \Upsilon^o = \begin{pmatrix} \mathrm{div} \\ \partial/\partial t \end{pmatrix}. \tag{5.27}$$

Using the Gauss–Green theorem and integration by parts, we have

$$\begin{aligned} \langle \Upsilon \boldsymbol{u} : \Upsilon^o \boldsymbol{u}^o \rangle &= \int_{\Omega_t} (\boldsymbol{u}_{,t} \, \mathrm{div}\, \boldsymbol{u}^o - (\nabla_{\mathrm{s}} \boldsymbol{u}) \cdot \boldsymbol{u}^o_{,t}) \, \mathrm{d}\Omega_t \\ &= \int_{\Omega_t} \boldsymbol{u} \left(-\frac{\partial}{\partial t} (\mathrm{div}\, \boldsymbol{u}^o) + \mathrm{div}\, \boldsymbol{u}^o_{,t} \right) \mathrm{d}\Omega_t + \int_{\Omega} \boldsymbol{u} \cdot (\mathrm{div}\, \boldsymbol{u}^o)|_{\partial I} \, \mathrm{d}\Omega \\ &\qquad - \int_I \oint_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{u}^o_{,t} \cdot \boldsymbol{n} \, \mathrm{d}\partial\Omega \, \mathrm{d}t. \end{aligned}$$

Thus, for a given admissible space $\mathcal{U}_{\mathrm{a}}$, the condition

$$\langle \Upsilon \boldsymbol{u} : \Upsilon^o \boldsymbol{u}^o \rangle = \langle \boldsymbol{u}, \Upsilon^* \Upsilon^o \boldsymbol{u}^o \rangle = 0 \quad \forall \boldsymbol{u} \in \mathcal{U}_{\mathrm{a}}$$

leads to the polar equilibrium condition in $\mathcal{U}^o_{\mathrm{a}} \subset \mathcal{U}^o$. For example, if we let $\mathcal{U}_{\mathrm{a}} = \{\boldsymbol{u} \in \mathcal{U} \mid \boldsymbol{u}(t,x) = 0 \ \forall t \in \partial I\}$, then $\mathcal{U}^o_{\mathrm{a}} = \{\boldsymbol{u}^o \in \mathcal{U}^o \mid \boldsymbol{n} \cdot \boldsymbol{u}^o_{,t}(t,x) = 0 \ \forall \boldsymbol{x} \in \partial\Omega\}$. Thus, on $\mathcal{U}_{\mathrm{a}}$ and $\mathcal{U}^o_{\mathrm{a}}$, we have $\partial^*_t = (\partial/\partial t)^* = -\partial/\partial t$, $\Lambda^* = \mathrm{grad}^* = -\,\mathrm{div}$, i.e.

$$\Upsilon^* = (-\partial/\partial t, \mathrm{div}), \qquad \Upsilon^{o*} = (-\,\mathrm{grad}, -\partial/\partial t), \tag{5.28}$$

and Υ^o is a null-source polar operator. For any given $\boldsymbol{u}^o \in \mathcal{U}^o_{\mathrm{a}}$, $\Upsilon^o \boldsymbol{u}^o$ is a homogeneous solution of the balance equation, i.e.

$$\Upsilon^* \Upsilon^o \boldsymbol{u}^o = \partial^*_t \,\mathrm{grad}^*\, \boldsymbol{u}^o - \mathrm{grad}^*\, \partial^*_t \boldsymbol{u}^o = \frac{\partial}{\partial t}(\mathrm{div}\, \boldsymbol{u}^o) - \mathrm{div}\left(\frac{\partial}{\partial t}\boldsymbol{u}^o\right) = 0 \quad \text{in } \Omega_t.$$

The particular solution of the balance equation $\Upsilon^* \boldsymbol{\varpi}^*_{\mathrm{p}} = DF(\boldsymbol{u})$ can be defined by $\boldsymbol{\varpi}^*_{\mathrm{p}} = (\bar{\boldsymbol{p}}, 0)^{\mathrm{T}}$, where $\bar{\boldsymbol{p}}$ is a particular solution of the non-homogeneous equation $\partial^*_t \bar{\boldsymbol{p}} = -DF(\boldsymbol{u})$. Thus, replacing $\boldsymbol{\varpi}^* = (\boldsymbol{p}, \boldsymbol{\sigma})^{\mathrm{T}}$ by $\Upsilon^o \boldsymbol{u}^o + \boldsymbol{\varpi}^*_{\mathrm{p}} = (\mathrm{div}\, \boldsymbol{u}^o + \bar{\boldsymbol{p}}, \boldsymbol{u}^o_{,t})^{\mathrm{T}}$, the polar variational problem for this geometrically linear dynamical system has a very simple form:

$$\Pi^o(\boldsymbol{u}^o) = \int_{\Omega_t} [U^*(\boldsymbol{u}^o_{,t}) - \tfrac{1}{2}\rho^{-1} |\nabla \cdot \boldsymbol{u}^o + \bar{\boldsymbol{p}}|^2] \,\mathrm{d}\Omega_t \to \text{sta} \quad \forall \boldsymbol{u}^o \in \mathcal{U}^o_k,$$

where $|*|$ stands for the Euclidean norm in $\mathbb{R}^3$. In linear elastodynamics such that the stored energy density $U(\boldsymbol{\epsilon}) = \frac{1}{2}\boldsymbol{\epsilon} : \boldsymbol{H} : \boldsymbol{\epsilon}$ is a quadratic function, where $\boldsymbol{H} = \{H_{ijkl}\}$ is a fourth-order symmetrical elastic tensor, satisfies $H_{ijkl} = H_{jikl} = H_{klij}$, the complementary energy density is simply defined as $U^*(\boldsymbol{\sigma}) = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{H}^{-1} : \boldsymbol{\sigma}$. In this case, the polar action Π^o is similar to Tabarrok's (1987) complementary action and Bui's (1992) dual formulation.

Clarify sentence?

Author: 'Tabarrok & Rimrott's (1987)'?

6. Complementary variational principle for contact mechanics

We now assume that the stored action $W : \mathcal{W} \to \mathbb{R}$ is a quadratic functional:

$$W(\boldsymbol{\varpi}) = \tfrac{1}{2}\langle \boldsymbol{\varpi} : \boldsymbol{C}\boldsymbol{\varpi}\rangle = \tfrac{1}{2}\langle \boldsymbol{v}, \rho \boldsymbol{v}\rangle - \tfrac{1}{2}\langle \boldsymbol{\epsilon}; \boldsymbol{H} : \boldsymbol{\epsilon}\rangle, \boldsymbol{C} = \begin{pmatrix} \rho & 0 \\ 0 & -\boldsymbol{H} \end{pmatrix},$$

where $\boldsymbol{C} : \mathcal{W} \to \mathcal{W}^*$ is a symmetrical operator. However, the external energy $F : \mathcal{U} \to \bar{\mathbb{R}}$ is a non-smooth extended canonical functional

$$F(\boldsymbol{u}) = \langle \boldsymbol{u}, \bar{\boldsymbol{u}}^* \rangle + J(\boldsymbol{u}), \tag{6.1}$$

where $J(\boldsymbol{u}) \in \Gamma_0(\mathcal{U})$ is a non-smooth super-potential.

For example, in frictional contact problems, the boundary is split into three parts: $\Gamma = \Gamma_{\mathrm{f}} \cup \Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{c}}$, where Γ_{c} is the contact surface. The external $F(\boldsymbol{u})$ can be written as

$$F(\boldsymbol{u}) = \int_{\Omega_t} \mathrm{e}^{\omega(\boldsymbol{x},t)} \boldsymbol{u} \cdot \bar{\boldsymbol{b}} \,\mathrm{d}\Omega_t + \int_{\Gamma_{\mathrm{f}}} \mathrm{e}^{\omega(\boldsymbol{x},t)} \boldsymbol{u} \cdot \bar{\boldsymbol{t}} \,\mathrm{d}\Gamma + \int_{\Gamma_{\mathrm{c}}} \mathrm{e}^{\omega(\boldsymbol{x},t)} j(\boldsymbol{u}) \,\mathrm{d}\Gamma, \tag{6.2}$$

where $j(\boldsymbol{u})$ is the frictional super-potential (cf., for example, Panagiotopoulos 1985). The duality relation $\boldsymbol{u}^* \in \partial F(\boldsymbol{u})$ gives the balance equation $\boldsymbol{u}^* = \bar{\boldsymbol{b}}$ in the domain Ω_t, the natural boundary condition on Γ_{f} and the contact condition on Γ_{c}:

$$\boldsymbol{u}^* \in \partial F(\boldsymbol{u}) = \begin{cases} \bar{\boldsymbol{t}} & \text{on } \Gamma_{\mathrm{f}}, \\ \partial j(\boldsymbol{u}) & \text{on } \Gamma_{\mathrm{c}}. \end{cases}$$

We further assume that the admissible displacement space $\mathcal{U}_{\rm a}$ is defined by (5.1). Thus, the primal problem takes the form

$$\Pi(\boldsymbol{u}) = \tfrac{1}{2}\langle \Upsilon \boldsymbol{u} : \boldsymbol{C}\Upsilon \boldsymbol{u}\rangle + F(\boldsymbol{u}) = \tfrac{1}{2}\langle \boldsymbol{u}, A\boldsymbol{u}\rangle + \langle \boldsymbol{u}, \bar{\boldsymbol{u}}^*\rangle + J(\boldsymbol{u}) \to \text{sta} \quad \forall \boldsymbol{u} \in \mathcal{U}_k,$$

where $A = \Upsilon^* C \Upsilon = \partial_t^* \rho \partial_t - \Lambda^* \boldsymbol{H} \Lambda$ is a self-adjoint operator. Since $A : \mathcal{U} \to \mathcal{U}^*$ is not monotone, $\Pi(\boldsymbol{u})$ may have many critical points. The primal feasible space in this case can be defined by $\mathcal{U}_k = \{\boldsymbol{u} \in \mathcal{U}_{\rm a} \mid \boldsymbol{u} \in \operatorname{dom} A \cap \operatorname{dom} J\}$. A critical point of Π is defined to be a point $\bar{\boldsymbol{u}} \in \mathcal{U}_k$ satisfying

$$0 \in A\bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}^* + \partial J(\bar{\boldsymbol{u}}). \tag{6.3}$$

The polar Lagrangian $\Xi^o : \mathcal{U}_{\rm a} \times \mathcal{U}_{\rm a}^o \to \mathbb{R}$ associated with this problem is

$$\Xi^o(\boldsymbol{u}, \boldsymbol{u}^o) = \langle \Upsilon \boldsymbol{u} : \Upsilon^o \boldsymbol{u}^o\rangle - \tfrac{1}{2}\langle \boldsymbol{C}^{-1}(\Upsilon^o \boldsymbol{u}^o + \boldsymbol{\varpi}_{\rm p}^*) : (\Upsilon^o \boldsymbol{u}^o + \boldsymbol{\varpi}_{\rm p}^*)\rangle + J(\boldsymbol{u}), \tag{6.4}$$

where $\boldsymbol{\varpi}_{\rm p}^* \in \mathcal{W}^*$ is a particular solution of the balance equation $\Upsilon^* \boldsymbol{\varpi}_{\rm p}^* + \bar{\boldsymbol{u}}^* = 0$. By definition, the polar action Π^o is obtained as

$$\begin{aligned} \Pi^o(\boldsymbol{u}^o) &= \operatorname{ext}\{\Xi^o(\boldsymbol{u}, \boldsymbol{u}^o) \mid \forall \boldsymbol{u} \in \mathcal{U}_{\rm a}\} \\ &= -\tfrac{1}{2}\langle \boldsymbol{C}^{-1}(\Upsilon^o \boldsymbol{u}^o + \boldsymbol{\varpi}_{\rm p}^*) : (\Upsilon^o \boldsymbol{u}^o + \boldsymbol{\varpi}_{\rm p}^*)\rangle - J^o(-\Upsilon^* \Upsilon^o \boldsymbol{u}^o), \end{aligned}$$

where $J^o : \mathcal{U}^o \to \bar{\mathbb{R}}$ is defined by

$$J^o(-\Upsilon^* \Upsilon^o \boldsymbol{u}^o) = \operatorname{ext}\{\langle \boldsymbol{u}, -\Upsilon^* \Upsilon^o \boldsymbol{u}^o\rangle - J(\boldsymbol{u}) \mid \forall \boldsymbol{u} \in \mathcal{U}_{\rm a}\}.$$

Thus, on the polar feasible space $\mathcal{U}_k^o$, the bi-polarity theorem holds. By the fact that J^o is a smooth functional over its effective domain, this polar variational problem should be easier than the original non-smooth primal problem.

Since $A = \Upsilon^* \boldsymbol{C} \Upsilon : \mathcal{U}_k \to \mathcal{U}^*$ is a self-adjoint operator, if $\mathcal{U}_k$ is a reflexive Banach space, we can let $\mathcal{U}^o = \mathcal{U}$ and choose the polar operator $\Upsilon^o = \boldsymbol{C}\Upsilon$. Thus, for a given particular solution $\boldsymbol{u}_{\rm p}$ satisfying $A\boldsymbol{u}_{\rm p} + \bar{\boldsymbol{u}}^* = 0$, replacing $\boldsymbol{u}^o$ by $\boldsymbol{u} - \boldsymbol{u}_{\rm p}$, and let $\boldsymbol{\varpi}_{\rm p}^* = \boldsymbol{C}\Upsilon \boldsymbol{u}_{\rm p}$, then Π^o can be written in the form

$$\Pi^o(\boldsymbol{u} - \boldsymbol{u}_{\rm p}) = -\tfrac{1}{2}\langle \boldsymbol{u}, A\boldsymbol{u}\rangle - J^o(-A(\boldsymbol{u} - \boldsymbol{u}_{\rm p})).$$

Let $\mathcal{U}_k^{\rm c} = \{\boldsymbol{u} \in \mathcal{U}_{\rm a} \mid A\boldsymbol{u} + \bar{\boldsymbol{u}}^* \in \operatorname{dom} J^o\}$ be the so-called *complementary feasible space*, and on which, we introduce the so-called *complementary action* $\Pi^{\rm c} : \mathcal{U} \to \bar{\mathbb{R}}$: Clarify sentence?

$$\Pi^{\rm c}(\boldsymbol{u}) = -\Pi^o(\boldsymbol{u} - \boldsymbol{u}_{\rm p}) = \tfrac{1}{2}\langle \boldsymbol{u}, A\boldsymbol{u}\rangle + J^o(-A\boldsymbol{u} - \bar{\boldsymbol{u}}^*). \tag{6.5}$$

Then, the complementary action principle can be proposed as

$$(\Pi^{\rm c}) : \Pi^{\rm c}(\boldsymbol{u}^{\rm c}) = \operatorname{sta} \Pi^{\rm c}(\boldsymbol{u}) \quad \forall \boldsymbol{u} \in \mathcal{U}_k^{\rm c}. \tag{6.6}$$

Since $J^o \in \Gamma_0(\mathcal{U}^*)$ is an extended canonical functional, the criticality condition of $\Pi^{\rm c}$ leads to the following complementary Euler–Lagrange inclusion:

$$0 \in A\boldsymbol{u}^{\rm c} + \partial[J^o \circ (-A)](\boldsymbol{u}^{\rm c} - \boldsymbol{u}_{\rm p}). \tag{6.7}$$

By the theory of convex analysis (cf., for example, Gao 2000*a*), we know that $\partial(J^o \circ A)(\boldsymbol{u}) = A^* \partial J^o(A\boldsymbol{u})$. Since $A = A^*$ is a self-adjoint operator, the complementary Euler–Lagrange inclusion can be written in the form

$$\boldsymbol{u}^{\rm c} \in \partial J^o(-A\boldsymbol{u}^{\rm c} - \bar{\boldsymbol{u}}^*). \tag{6.8}$$

Theorem 6.1 (complementary theorem). *Let $\mathcal{N}(A) = \{\boldsymbol{u}_o \in \mathcal{U}_k \mid A\boldsymbol{u}_o = 0\}$ be the null-space of the operator A.*

(T1) *If $\bar{\boldsymbol{u}}$ is a critical point of Π, then for any given $\boldsymbol{u}_o \in \mathcal{N}(A)$, $\boldsymbol{u}^{\mathrm{c}} = \boldsymbol{u}_o + \bar{\boldsymbol{u}}$ is a critical point of Π^{c}.*

(T2) *Conversely, if $\mathcal{U}_k^{\mathrm{c}} \neq \emptyset$ and if $\boldsymbol{u}^{\mathrm{c}}$ is a critical point of Π^{c}, then $\bar{\boldsymbol{u}} = \boldsymbol{u}_o + \boldsymbol{u}^{\mathrm{c}}$ is a critical point of Π for some given $\boldsymbol{u}_o \in \mathcal{N}(A)$.*

(T3) *If $\mathcal{U}_k^{\mathrm{c}} \neq \emptyset$, $\bar{\boldsymbol{u}}$ and $\boldsymbol{u}^{\mathrm{c}}$ are critical points of Π and Π^{c}, respectively, then the complementary conditions*

$$\left.\begin{aligned} \Pi(\bar{\boldsymbol{u}}) + \Pi^{\mathrm{c}}(\bar{\boldsymbol{u}} + \boldsymbol{u}_o) = 0 \\ \Pi(\boldsymbol{u}_o + \boldsymbol{u}^{\mathrm{c}}) + \Pi^{\mathrm{c}}(\boldsymbol{u}^{\mathrm{c}}) = 0 \end{aligned}\right\} \quad \forall \boldsymbol{u}_o \in \mathcal{N}(A) \tag{6.9}$$

hold.

Proof.

(T1) If $\bar{\boldsymbol{u}}$ is a critical point of Π, then $0 \in A\bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}^* + \partial J(\bar{\boldsymbol{u}})$. By the Fenchel duality formula, this is equivalent to $\bar{\boldsymbol{u}} \in \partial J^o(-A\bar{\boldsymbol{u}} - \bar{\boldsymbol{u}}^*)$. This means that $\bar{\boldsymbol{u}}$ is a critical point of Π^{c}. Since $\Pi^{\mathrm{c}}(\boldsymbol{u})$ is invariant under translation in $\mathcal{N}(A)$, i.e. for any given $\boldsymbol{u}_o \in \mathcal{N}(A)$, $\Pi^{\mathrm{c}}(\boldsymbol{u}+\boldsymbol{u}_o) = \Pi^{\mathrm{c}}(\boldsymbol{u})\ \forall \boldsymbol{u} \in \mathcal{U}_k^{\mathrm{c}}$, then all the points in $\bar{\boldsymbol{u}} + \mathcal{N}(A)$ are also critical points of Π^{c}.

(T2) Conversely, let $\boldsymbol{u}^{\mathrm{c}}$ be a critical point of Π^{c}. Since $\mathcal{U}_k^{\mathrm{c}}$ is not empty, then, for any given $\boldsymbol{u}_o \in \mathcal{N}(A)$, we have $0 \in A(\boldsymbol{u}^{\mathrm{c}} + \boldsymbol{u}_o) + \partial[J^o \circ (-A)](\boldsymbol{u}^{\mathrm{c}} + \boldsymbol{u}_o - \boldsymbol{u}_{\mathrm{p}})$, i.e. $\boldsymbol{u}^{\mathrm{c}} + \boldsymbol{u}_o \in \partial J^o(-A(\boldsymbol{u}^{\mathrm{c}} + \boldsymbol{u}_o) - \bar{\boldsymbol{u}}^*)$. By the Fenchel duality we know that this is equivalent to $-A(\boldsymbol{u}^{\mathrm{c}} + \boldsymbol{u}_o) - \bar{\boldsymbol{u}}^* \in \partial J(\boldsymbol{u}^{\mathrm{c}} + \boldsymbol{u}_o)$, or $0 \in A(\boldsymbol{u}^{\mathrm{c}} + \boldsymbol{u}_o) + \bar{\boldsymbol{u}}^* + \partial J(\boldsymbol{u}^{\mathrm{c}} + \boldsymbol{u}_o)$, since $A\boldsymbol{u}_{\mathrm{p}} + \bar{\boldsymbol{u}}^* = 0$. This shows that $\bar{\boldsymbol{u}} = \boldsymbol{u}^{\mathrm{c}} + \boldsymbol{u}_o$ is indeed a critical point of Π for any given $\boldsymbol{u}_o \in \mathcal{N}(A)$.

(T3) This statement follows from the bi-polarity theorem.

■

In the case where $F(\boldsymbol{u}) = \langle \boldsymbol{u}, \bar{\boldsymbol{u}}^* \rangle + J(\boldsymbol{u})$ is a non-smooth convex functional, defined by (6.2), then, for homogeneous boundary conditions, the total action is a non-smooth DC functional

$$\begin{aligned} \Pi(\boldsymbol{u}) &= \tfrac{1}{2}\langle \boldsymbol{u}, A\boldsymbol{u} \rangle + \langle \boldsymbol{u}, \bar{\boldsymbol{u}}^* \rangle + J(\boldsymbol{u}) \\ &= \int_{\Omega_t} \tfrac{1}{2}[\rho \boldsymbol{u}_{,t} \cdot \boldsymbol{u}_{,t} - (\nabla \boldsymbol{u})^{\mathrm{T}} : \boldsymbol{H} : (\nabla \boldsymbol{u}) + \boldsymbol{u} \cdot \bar{\boldsymbol{b}}]\, \mathrm{d}\Omega_t + \int_{\Gamma_{\mathrm{f}}} \boldsymbol{u} \cdot \bar{\boldsymbol{t}}\, \mathrm{d}\Gamma + \int_{\Gamma_{\mathrm{c}}} j(\boldsymbol{u})\, \mathrm{d}\Gamma, \end{aligned}$$

Author: first parenthesis ')' on second line replaced with angled bracket '⟩' – OK?

where $A : \mathcal{U}_k \to \mathcal{U}^*$ is a self-adjoint wave operator:

$$A\boldsymbol{u} = \begin{cases} -\dfrac{\partial}{\partial t}\left(\rho \dfrac{\partial}{\partial t}\boldsymbol{u}\right) + \operatorname{div} \boldsymbol{H} \operatorname{grad} \boldsymbol{u} & \text{in } \Omega_t, \\ \boldsymbol{n} \cdot (\boldsymbol{H} : (\nabla \boldsymbol{u})) & \text{on } \partial\Omega. \end{cases}$$

On the complementary feasible space $\mathcal{U}_k^{\mathrm{c}}$, the complementary action has a very simple form:

$$\Pi^{\mathrm{c}}(\boldsymbol{u}) = \tfrac{1}{2}\langle \boldsymbol{u}, A\boldsymbol{u}\rangle + J^{o}(-A\boldsymbol{u} - \bar{\boldsymbol{u}}^*)$$
$$= \int_{\Omega_t} \tfrac{1}{2}[\rho \boldsymbol{u}_{,t} \cdot \boldsymbol{u}_{,t} - (\nabla \boldsymbol{u})^{\mathrm{T}} : \boldsymbol{H} : (\nabla \boldsymbol{u})]\,\mathrm{d}\Omega_t + \int_{\Gamma_{\mathrm{c}}} j^{o}(-\boldsymbol{n} \cdot (\boldsymbol{H} : \nabla \boldsymbol{u}))\,\mathrm{d}\Gamma.$$

As shown in Gao (2001*a*) that if the super-potential $j(\boldsymbol{u})$ is non-smooth on $\mathcal{U}_k$, its conjugate $j^{o}(\boldsymbol{u}^*)$, however, could be a smooth functional. The complementary variational problem will then be much easier than the primal problem.

Clarify sentence?

7. Concluding remarks

Duality and triality, as well as associated concepts—such as complementarity, polarity, symmetry and symmetry breaking—play more and more important roles in modern mathematics and science. The inner beauty of these concepts owes much to the fact that many different natural phenomena can be put in a unified trio-canonical framework (cf., for example, Gao 2000*a*). By the fact that the canonical physical variables always appear in pairs, the canonical dual transformation method can be used to solve many problems in natural systems. The associated extended Lagrange duality and polar Hamiltonian may possess profound computational impacts. As we can see, for convex systems the bi-polarity and complementary theorems take particular symmetrical forms, which convert non-smooth primal problems into smooth polar or complementary problems. For non-convex Newtonian dynamical systems, as long as there exists a geometrical operator Λ such that the tri-canonical forms can be characterized correctly, the canonical dual transformation method can be used to establish nice complementary-dual formulations. The interesting triality theorems reveal the intrinsic symmetry in non-convex systems, which can be used to develop efficient alternative algorithms for robust computations. Due to the page limit, the applications in non-convex dynamics of the triality theory will be given in another paper, where it shows that the chaotical trajectories of the Duffing-type system form an invariant set in canonical dual phase space. This invariant set plays a key role in stability analysis of non-convex Hamilton systems and in feedback controlling against chaos (Gao 2001*b*). By using the so-called sequential canonical dual transformation method (see Gao 2000*c*), the results presented in this paper can be generalized to non-Newtonian systems, where the time operator ∂_t is nonlinear, as well as non-convex systems with multiple potential wells.

OK?

Author: what other paper?

OK?

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