

DUALITY THEORY: II. BI-DUALITY IN NONCONVEX OPTIMIZATION

It is known that in convex optimization, the Lagrangian associated with a constrained problem is usually a saddle function, which leads to the classical *saddle Lagrange duality* (i.e. the *mono-duality*) theory. In nonconvex optimization, a so-called super-Lagrangian was introduced in [1], which leads to a nice bi-duality theory in convex *Hamiltonian systems* and in the so-called *d.c. programming*.

Super-Lagrangian Duality.

Definition 1 Let $L(x, y^*)$ be an arbitrary given real-valued function on $\mathcal{X} \times \mathcal{Y}^*$.

A function $L : \mathcal{X} \times \mathcal{Y}^* \rightarrow \mathbb{R}$ is said to be a *super-critical function* (or ∂^+ *function*) on $\mathcal{X} \times \mathcal{Y}^*$ if it is concave in each of its arguments.

A function $L : \mathcal{X} \times \mathcal{Y}^* \rightarrow \mathbb{R}$ is said to be a *sub-critical function* (or ∂^- *function*) on $\mathcal{X} \times \mathcal{Y}^*$ if $-L$ is a super-critical function on $\mathcal{X} \times \mathcal{Y}^*$.

A point (\bar{x}, \bar{y}^*) is said to be a *super-critical point* (or ∂^+ -*critical point*) of L on $\mathcal{X} \times \mathcal{Y}^*$ if

$$L(\bar{x}, y^*) \leq L(\bar{x}, \bar{y}^*) \geq L(x, \bar{y}^*) \quad (1)$$

holds for all $(x, y^*) \in \mathcal{X} \times \mathcal{Y}^*$.

A point (\bar{x}, \bar{y}^*) is said to be a *sub-critical point* (or ∂^- -*critical point*) of L on $\mathcal{X} \times \mathcal{Y}^*$ if

$$L(\bar{x}, y^*) \geq L(\bar{x}, \bar{y}^*) \leq L(x, \bar{y}^*) \quad (2)$$

holds for all $(x, y^*) \in \mathcal{X} \times \mathcal{Y}^*$. \diamond

Clearly, a point (\bar{x}, \bar{y}^*) is a sub-critical point of L on $\mathcal{X} \times \mathcal{Y}^*$ if and only if it is a super-critical

saddle Lagrange duality \rightarrow *saddle Lagrange duality*

mono-duality \rightarrow *mono-duality*

Hamiltonian systems \rightarrow *Hamilton system*

d.c. programming \rightarrow *d.c. programming*

super-critical function \rightarrow *super-critical function*

∂^+ *function* \rightarrow ∂^+ -*function*

sub-critical function \rightarrow *sub-critical function*

∂^- *function* \rightarrow ∂^- -*function*

super-critical point \rightarrow *super-critical point*

∂^+ -*critical point* \rightarrow ∂^+ -*critical point*

sub-critical point \rightarrow *sub-critical point*

∂^- -*critical point* \rightarrow ∂^- -*critical point*

super-Lagrangian \rightarrow *super-Lagrangian*

Lagrange form \rightarrow *Lagrange form*

sub-Lagrangian \rightarrow *sub-Lagrangian*

point of $-L$ on $\mathcal{X} \times \mathcal{Y}^*$. A super-critical function $L(x, y^*)$ is called the *super-Lagrangian* if it is a *Lagrange form* associated with a constrained optimization problem. $L(x, y^*)$ is called the *sub-Lagrangian* if $-L(x, y^*)$ is a super-Lagrangian.

For example, the quadratic function

$$L(x, y) = axy - \frac{1}{2}bx^2 - \frac{1}{2}cy^2, \quad b, c > 0$$

is concave for each x , and y , and hence, is a super-critical point function on $\mathbb{R} \times \mathbb{R}$. But $L(x, y)$ is not concave on the vector (x, y) since the Hessian matrix of L

$$D^2L(x, y) = \begin{pmatrix} -b & a \\ a & -c \end{pmatrix},$$

is not necessarily to be negative-definite for any $a \in \mathbb{R}$ and $b, c > 0$. L is a sub-critical function if $b, c < 0$. But L may not be convex on (x, y) for the same reason.

Since L is a sub-Lagrangian if and only if $-L$ is a super-Lagrangian, here we only consider the duality theory for the super-Lagrangian.

Theorem 1 (*Super-Critical Point*)

Let $L(x, y^*)$ be an arbitrary given function, partially Gâteaux differentiable on an open subset $\mathcal{X}_a \times \mathcal{Y}_a^* \subset \mathcal{X} \times \mathcal{Y}^*$. If $(\bar{x}, \bar{y}^*) \in \mathcal{X}_a \times \mathcal{Y}_a^*$ is either a super-critical or sub-critical point of L , then (\bar{x}, \bar{y}^*) is a critical point of L on $\mathcal{X}_a \times \mathcal{Y}_a^*$.

Any critical point of a Gâteaux differentiable super-Lagrangian is a super-critical point. However, if (\bar{x}, \bar{y}^*) is a super-critical point of L , it does not follow that L is a super-Lagrangian. In the d.c. programming or variational analysis of convex Hamiltonian systems, the following

statements are of important theoretical value.

(S1) Under certain necessary and sufficient conditions we have

$$\inf_{x \in \mathcal{X}_a} \sup_{y^* \in \mathcal{Y}_a^*} L(x, y^*) = \inf_{y^* \in \mathcal{Y}_a^*} \sup_{x \in \mathcal{X}_a} L(x, y^*). \quad (3)$$

A statement of this type is called a *super-minimax theorem* and the pair (\bar{x}, \bar{y}^*) is called a *super-minimax point* of L on $\mathcal{X}_a \times \mathcal{Y}_a^*$.

(S2) Under certain conditions, a pair $(\bar{x}, \bar{y}^*) \in \mathcal{X}_a \times \mathcal{Y}_a^*$ exists such that

$$L(x, \bar{y}^*) \leq L(\bar{x}, \bar{y}^*) \geq L(\bar{x}, y^*) \quad (4)$$

holds for all $(x, y^*) \in \mathcal{X}_a \times \mathcal{Y}_a^*$. A statement of this type is called a *super-critical point theorem*.

By the fact that the maxima of $L(x, y^*)$ can be taken in either order on $\mathcal{X}_a \times \mathcal{Y}_a^*$, the equality

$$\sup_{x \in \mathcal{X}_a} \sup_{y^* \in \mathcal{Y}_a^*} L(x, y^*) = \sup_{y^* \in \mathcal{Y}_a^*} \sup_{x \in \mathcal{X}_a} L(x, y^*) \quad (5)$$

always holds. A pair (\bar{x}, \bar{y}^*) which maximizes L on $\mathcal{X}_a \times \mathcal{Y}_a^*$ is called a *super-maximum point* of L on $\mathcal{X}_a \times \mathcal{Y}_a^*$.

For a given super-Lagrangian $L : \mathcal{X}_a \times \mathcal{Y}_a^* \rightarrow \mathbb{R}$, we let $\mathcal{X}_k \subseteq \mathcal{X}_a$ and $\mathcal{Y}_s^* \subseteq \mathcal{Y}_a^*$ such that

$$\begin{aligned} \sup_{y^* \in \mathcal{Y}_a^*} L(x, y^*) &< +\infty \quad \forall x \in \mathcal{X}_k, \\ \sup_{x \in \mathcal{X}_a} L(x, y^*) &< +\infty \quad \forall y^* \in \mathcal{Y}_s^*. \end{aligned}$$

Theorem 2 (Super-Lagrangian Duality)

Let the Lagrangian $L : \mathcal{X} \times \mathcal{Y}^* \rightarrow \mathbb{R}$ be an arbitrary given function. If there exists either a *super-maximum point* $(\bar{x}, \bar{y}^*) \in \mathcal{X}_a \times \mathcal{Y}_a^*$ such that

$$\begin{aligned} L(\bar{x}, \bar{y}^*) &= \max_{x \in \mathcal{X}_a} \max_{y^* \in \mathcal{Y}_a^*} L(x, y^*) \\ &= \max_{y^* \in \mathcal{Y}_a^*} \max_{x \in \mathcal{X}_a} L(x, y^*), \end{aligned} \quad (6)$$

super-minimax theorem \rightarrow *super-minimax theorem*
super-minimax point \rightarrow *super-minimax point*
super-critical point theorem \rightarrow *super-critical point theorem*
super-maximum point \rightarrow *super-maximum point*
primal feasible set \rightarrow *primal feasible set*

or a *super-minimax point* $(\bar{x}, \bar{y}^*) \in \mathcal{X}_a \times \mathcal{Y}_a^*$ such that

$$\begin{aligned} L(\bar{x}, \bar{y}^*) &= \min_{x \in \mathcal{X}_a} \max_{y^* \in \mathcal{Y}_a^*} L(x, y^*) \\ &= \min_{y^* \in \mathcal{Y}_a^*} \max_{x \in \mathcal{X}_a} L(x, y^*), \end{aligned} \quad (7)$$

then (\bar{x}, \bar{y}^*) is a *super-critical point* of L on $\mathcal{X}_a \times \mathcal{Y}_a^*$.

Conversely, if L is partially Gâteaux differentiable on an open subset $\mathcal{X}_a \times \mathcal{Y}_a^* \subset \mathcal{X} \times \mathcal{Y}^*$, and (\bar{x}, \bar{y}^*) is a *super-critical point* of L on $\mathcal{X}_a \times \mathcal{Y}_a^*$, then either the *super-maximum theorem* in the form

$$\begin{aligned} L(\bar{x}, \bar{y}^*) &= \max_{x \in \mathcal{X}_k} \max_{p \in \mathcal{Y}_a^*} L(x, y^*) \\ &= \max_{y^* \in \mathcal{Y}_s^*} \max_{x \in \mathcal{X}_a} L(x, y^*), \end{aligned} \quad (8)$$

holds, or the *super-minimax theorem* in the form

$$\begin{aligned} L(\bar{x}, \bar{y}^*) &= \min_{x \in \mathcal{X}_k} \max_{y^* \in \mathcal{Y}_a^*} L(x, y^*) \\ &= \min_{y^* \in \mathcal{Y}_s^*} \max_{x \in \mathcal{X}_a} L(x, y^*) \end{aligned} \quad (9)$$

holds.

This super-Lagrangian duality theorem shows a very important fact in Hamiltonian systems, i.e. the critical points of the Lagrangian L either maximize or minimaximize L on $\mathcal{X}_k \times \mathcal{Y}_s^*$ in either order.

Nonconvex Primal and Dual Problems.

Let $L : \mathcal{X}_a \times \mathcal{Y}_a^* \rightarrow \mathbb{R}$ be an arbitrary given super-critical function. For any fixed $x \in \mathcal{X}_a$, let

$$P(x) = \sup_{y^* \in \mathcal{Y}_a^*} L(x, y^*). \quad (10)$$

Clearly, the function $P(x)$ may not be either convex or concave. Let $\mathcal{X}_k \subset \mathcal{X}_a$ be the *primal feasible set* such that $P : \mathcal{X}_k \rightarrow \mathbb{R}$ is finite and Gâteaux differentiable. Then for nonconvex function P , two primal problems can be proposed as

$$(\mathcal{P}_{\text{inf}}) : P(x) \rightarrow \min \quad \forall u \in \mathcal{X}_k \quad (11)$$

$$(\mathcal{P}_{\text{sup}}) : P(x) \rightarrow \max \quad \forall u \in \mathcal{X}_k \quad (12)$$

The problems $(\mathcal{P}_{\text{inf}})$ and $(\mathcal{P}_{\text{sup}})$ are realisable if the primal feasible set \mathcal{X}_k is not empty.

Dually, for any fixed $y^* \in \mathcal{Y}_a^*$, let

$$P^d(y^*) = \sup_{x \in \mathcal{X}_a} L(x, y^*) \quad (13)$$

with the *dual feasible set* $\mathcal{Y}_s^* \subset \mathcal{Y}_a^*$ such that $P^d : \mathcal{Y}_s^* \rightarrow \mathbb{R}$ is finite and Gâteaux differentiable. The two nonconvex dual problems are

$$(\mathcal{P}_{\text{inf}}^d) : P^d(y^*) \rightarrow \min \forall y^* \in \mathcal{Y}_s^*, \quad (14)$$

$$(\mathcal{P}_{\text{sup}}^d) : P^d(y^*) \rightarrow \max \forall y^* \in \mathcal{Y}_s^*. \quad (15)$$

These two dual problems are realisable if the dual feasible set \mathcal{Y}_s^* is not empty.

Theorem 3 (*Bi-Duality Theorem*)

Let $L : \mathcal{X}_a \times \mathcal{Y}_a^* \rightarrow \mathbb{R}$ be a given arbitrary function such that P and P^d are well-defined by (10) and (13) on the open subsets \mathcal{X}_k and \mathcal{Y}_s^* , respectively.

(1) If (\bar{x}, \bar{y}^*) is a super-critical point of L on $\mathcal{X}_k \times \mathcal{Y}_s^*$, then $DP(\bar{x}) = 0$, $DP^d(\bar{y}^*) = 0$, and

$$P(\bar{x}) = L(\bar{x}, \bar{y}^*) = P^d(\bar{y}^*). \quad (16)$$

(2) If (\bar{x}, \bar{y}^*) is a super-critical point of L on $\mathcal{X}_k \times \mathcal{Y}_s^*$, then \bar{x} is a minimizer of P on \mathcal{X}_k if and only if \bar{y}^* is a minimizer of P^d on \mathcal{Y}_s^* , i.e. the double-min duality

$$P(\bar{x}) = \inf_{x \in \mathcal{X}_k} P(x) \Leftrightarrow \inf_{y^* \in \mathcal{Y}_s^*} P^d(y^*) = P^d(\bar{y}^*) \quad (17)$$

holds.

(3) If (\bar{x}, \bar{y}^*) is a super-critical point of L on $\mathcal{X}_k \times \mathcal{Y}_s^*$, then \bar{x} is a maximizer of P on \mathcal{X}_k if and only if \bar{y}^* is a maximizer of P^d on \mathcal{Y}_s^* , i.e. the double-max duality

$$P(\bar{x}) = \sup_{x \in \mathcal{X}_k} P(x) \Leftrightarrow \sup_{y^* \in \mathcal{Y}_s^*} P^d(y^*) = P^d(\bar{y}^*) \quad (18)$$

holds.

D.C. Programming and Hamiltonian.

In d.c. programming, the primal function $P : \mathcal{X}_k \rightarrow \mathbb{R}$ can be written as

$$P(x) = W(\Lambda x) - F(x),$$

dual feasible set \rightarrow *dual feasible set*

double-min duality \rightarrow *double-min duality*

double-max duality \rightarrow *double-max duality*

Legendre duality relations \rightarrow *Legendre duality relations*

Legendre conjugates \rightarrow *Legendre conjugate*

total action \rightarrow *total action*

Lagrangian form \rightarrow *Lagrange form*

where $\Lambda : \mathcal{X} \rightarrow \mathcal{Y}$ is a linear operator, $W : \mathcal{Y}_a \rightarrow \mathbb{R}$ and $F : \mathcal{X}_a \rightarrow \mathbb{R}$ are two convex, Gâteaux differentiable real-valued functions, satisfying the *Legendre duality relations*

$$x^* = DF(x) \Leftrightarrow x = DF^*(x^*) \Leftrightarrow$$

$$\langle x, x^* \rangle = F(x) + F^*(x^*)$$

on $\mathcal{X}_a \times \mathcal{X}_a^*$, and

$$y^* = DW(y) \Leftrightarrow y = DW^*(y^*) \Leftrightarrow$$

$$\langle y, y^* \rangle = W(y) + W^*(y^*)$$

on $\mathcal{Y}_a \times \mathcal{Y}_a^*$, where $F^* : \mathcal{X}_a^* \rightarrow \mathbb{R}$ and $W^* : \mathcal{Y}_a^* \rightarrow \mathbb{R}$ are the *Legendre conjugates* of F and W , respectively.

In dynamical systems, if $\Lambda = d/dt$ is a differential operator, its adjoint associated with the standard bilinear forms in \mathcal{L}^2 is $\Lambda^* = -d/dt$. If W denotes the kinetic energy, F stands for potential energy, then the primal function $P(x) = W(\Lambda x) - F(x)$ is the *total action* of the system. The primal feasible set $\mathcal{X}_k \subset \mathcal{X}$, defined by

$$\mathcal{X}_k = \{x \in \mathcal{X}_a \mid \Lambda x \in \mathcal{Y}_a\},$$

is called the kinetically admissible space. Clearly, $P : \mathcal{X}_k \rightarrow \mathbb{R}$ is nonconvex.

The *Lagrangian form* associated with the nonconvex primal problems is defined by

$$L(x, y^*) = \langle \Lambda x; y^* \rangle - W^*(y^*) - F(x), \quad (19)$$

which is Gâteaux differentiable on $\mathcal{X}_a \times \mathcal{Y}_a^*$. The critical condition $DL(\bar{x}, \bar{y}^*) = 0$ leads to the Lagrange equations:

$$\Lambda \bar{x} = DW^*(\bar{y}^*), \quad \Lambda^* \bar{y}^* = DF(\bar{x}).$$

Clearly, $L : \mathcal{X}_a \times \mathcal{Y}_a^* \rightarrow \mathbb{R}$ is a super-critical function, and

$$P(x) = \sup_{y^* \in \mathcal{Y}_a^*} L(x, y^*) \quad \forall x \in \mathcal{X}_k.$$

Dually, for any given dual feasible $y^* \in \mathcal{Y}_s^*$,

$$P^d(y^*) = \sup_{x \in \mathcal{X}_a} L(x, y^*) = F^*(\Lambda^* y^*) - W^*(y^*),$$

where the dual feasible set $\mathcal{Y}_s^* \subset \mathcal{Y}_a^*$ is defined by

$$\mathcal{Y}_s^* = \{y^* \in \mathcal{Y}_a^* \mid \Lambda^* y^* \in \mathcal{X}_a^*\}.$$

The criticality conditions $DL(\bar{x}, \bar{y}^*) = 0$, $DP(\bar{x}) = 0$ and $DP^d(\bar{y}^*) = 0$ are equivalent to each other.

The Hamiltonian $H : \mathcal{X}_a \times \mathcal{Y}_a^* \rightarrow \mathbb{R}$ associated with the Lagrangian L is defined by

$$\begin{aligned} H(x, y^*) &= \langle \Lambda x; y^* \rangle - L(x, y^*) \\ &= W^*(y^*) + F(x). \end{aligned} \quad (20)$$

For d.c. programming, $H(x, y^*)$ is a convex function on $\mathcal{X}_a \times \mathcal{Y}_a^*$. The critical point (\bar{x}, \bar{y}^*) of L satisfies the Hamiltonian *canonical form*

$$\Lambda \bar{x} = D_{y^*} H(\bar{x}, \bar{y}^*), \quad \Lambda^* \bar{y}^* = D_x H(\bar{x}, \bar{y}^*).$$

Particularly, if $W(\Lambda x) = \frac{1}{2} \langle \Lambda x, C \Lambda x \rangle$ is a quadratic function, $C : \mathcal{Y}_a \rightarrow \mathcal{Y}_a^*$ is a symmetric operator such that the composite operator $A = \Lambda^* C \Lambda = A^*$ is self adjoint, then the total action can be written as

$$P(x) = \frac{1}{2} \langle x, Ax \rangle - F(x).$$

Let $P^c(x) = -P^d(C \Lambda x)$, then the function $P^c : \mathcal{X}_a \rightarrow \mathbb{R}$

$$P^c(x) = \frac{1}{2} \langle x, Ax \rangle - F^*(Ax)$$

is the so-called *Clarke dual action* (see [1]).

Theorem 4 (Clarke Duality Theorem)

Let $A : \mathcal{X}_k \subset \mathcal{X}_a \rightarrow \mathcal{X}_a^*$ be a closed, self-adjoint operator, and $\text{Ker } A = \{x \in \mathcal{X} \mid Ax = 0 \in \mathcal{X}^*\}$ the null space of A . If $\bar{x} \in \mathcal{X}_k$ is a critical point of P , then any vector $x \in \text{Ker } A + \bar{x}$ is a critical point of P^c .

Conversely, if there exists a $x_o \in \mathcal{X}_k$ such that $Ax_o \in \mathcal{X}_a^*$, then for a given critical point \bar{x} of P^c , any vector $x \in \text{Ker } A + \bar{x}$ is a critical point of P .

Example. Let us consider a very simple one-dimensional optimization problem with constraint

$$\begin{aligned} F(x) &= \frac{1}{2} k x^2 - f x \rightarrow \max \\ \text{s.t. } & a \leq x \leq b, \end{aligned} \quad (21)$$

where $k > 0$ and $f \in \mathbb{R}$ are given constants. We assume that $-\infty < a < 0 < b < \infty$. Since $F(x)$ is strictly convex on the closed set $[a, b]$, the maximum is attained only on the boundary, i.e.

$$\sup_{x \in [a, b]} F(x) = \max\{F(a), F(b)\} < \infty.$$

The classical Lagrange multiplier method cannot be used for this nonconvex problem. To set this problem within our framework, we need only set $\mathcal{X} = \mathbb{R}$, $\mathcal{X}_a = [a, b]$ and let $\Lambda = 1$, so that

$$y = \Lambda x = x \in \mathcal{Y} = \mathbb{R}.$$

Thus, the range of the mapping $\Lambda : \mathcal{X}_a \rightarrow \mathcal{Y} = \mathbb{R}$ is $\mathcal{Y}_a = [a, b]$. Let

$$W(y) = \begin{cases} 0 & \text{if } y \in \mathcal{Y}_a, \\ +\infty & \text{if } y \notin \mathcal{Y}_a. \end{cases}$$

It is not difficult to check that $W : \mathcal{Y} \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex. On \mathcal{Y}_a , W is finite and differentiable. Thus, the primal feasible space can be defined by

$$\mathcal{X}_k = \{x \in \mathcal{X}_a \mid \Lambda x = x \in \mathcal{Y}_a\} = [a, b].$$

Clearly, on \mathcal{X}_k $P(x) = W(\Lambda x) - F(x) = F(x)$. The constrained maximization problem (21) is then equivalent to the standard nonconvex minimization problem $(\mathcal{P}_{\text{inf}}) : P(x) \rightarrow \min \quad \forall x \in \mathcal{X}_k$.

Since $F(x)$ is strictly convex and differentiable on $\mathcal{X}_a = [a, b]$, and

$$x^* = DF(x) = kx - f \in \mathcal{X}_a^*$$

is invertible, where

$$\mathcal{X}_a^* = [ak - f, bk - f] \subset \mathcal{X}^* = \mathbb{R}$$

the Legendre conjugate $P^c : \mathcal{X}_a^* \rightarrow \mathbb{R}$ can easily be obtained as

$$F^*(x^*) = \max_{x \in \mathcal{X}_a} \{xx^* - F(x)\} = \frac{1}{2k}(x^* + f)^2.$$

By the *Fenchel transformation*, the conjugate of the nonsmooth function W can be obtained

canonical form \rightarrow *canonical form*

Clarke dual action \rightarrow *Clarke dual action*

Fenchel transformation \rightarrow *Fenchel transformation*

as

$$\begin{aligned} W^*(y^*) &= \sup_{y \in \mathcal{Y}} \{yy^* - W(y)\} = \max_{y \in \mathcal{Y}_a} yy^* \\ &= \begin{cases} by^* & \text{if } y^* > 0, \\ 0 & \text{if } y^* = 0, \\ ay^* & \text{if } y^* < 0. \end{cases} \end{aligned}$$

It is convex and differentiable on $\mathcal{Y}_a^* = \mathcal{Y}^* = \mathbb{R}$.

On $\mathcal{X}_a \times \mathcal{Y}_a^* = [a, b] \times \mathbb{R}$, the Lagrange form for this nonconvex programming is well-defined by

$$\begin{aligned} L(x, y^*) &= y^* \Lambda x - W^*(y^*) - F(x) \\ &= \begin{cases} xy^* - by^* - \frac{1}{2}kx^2 + fx & \text{if } y^* \geq 0, \\ xy^* - ay^* - \frac{1}{2}kx^2 + fx & \text{if } y^* < 0. \end{cases} \end{aligned}$$

Since both W^* and F are convex, $L(x, y^*)$ is a super-critical point function. If $x \in \mathcal{X}_k = [a, b]$, then

$$P(x) = \sup_{y^* \in \mathcal{Y}_a^*} L(x, y^*).$$

On the other hand, for any y^* in the dual feasible space

$$\begin{aligned} \mathcal{Y}_s^* &= \{y^* \in \mathcal{Y}_a^* = \mathbb{R} \mid \Lambda^* y^* = y^* \in \mathcal{X}_a^*\} \\ &= [ak - f, bk - f], \end{aligned}$$

the dual function is obtained by

$$\begin{aligned} P^d(y^*) &= \sup_{x \in \mathcal{X}_a} L(x, y^*) \\ &= \sup_{x \in \mathbb{R}} \{(\Lambda x)y^* - F(x)\} - W^*(y^*) \\ &= F^*(\Lambda^* y^*) - W^*(y^*), \end{aligned}$$

where

$$\begin{aligned} F^*(\Lambda^* y^*) &= \sup_{x \in \mathcal{X}_a} \{(\Lambda x)y^* - F(x)\} \\ &= \sup_{x \in \mathbb{R}} \{x(y + f) - \frac{1}{2}kx^2\} \\ &= \frac{1}{2k}(y^* + f)^2. \end{aligned}$$

Thus, the dual action P^d is well defined on \mathcal{Y}_s^* by

$$P^d(y^*) = \begin{cases} \frac{1}{2k}(y^* + f)^2 - by^* & \text{if } y^* > 0, \\ \frac{1}{2k}f^2 & \text{if } y^* = 0, \\ \frac{1}{2k}(y^* + f)^2 - ay^* & \text{if } y^* < 0. \end{cases} \quad (22)$$

This is a double-well function on \mathbb{R} (see Fig. 1). The dual problem

$$(P_{\text{inf}}^d) : P^d(y^*) \rightarrow \min \quad \forall y^* \in \mathcal{Y}_s^*$$

is a convex optimization problem on either

$$\mathcal{Y}_s^{*+} = \{y^* \in \mathcal{Y}_s^* \mid y^* > 0\}$$

or

$$\mathcal{Y}_s^{*-} = \{y^* \in \mathcal{Y}_s^* \mid y^* < 0\}.$$

In n -dimensional problems, this dual problem is much easier than the primal problem. The criticality condition leads to

$$\bar{y}^* = \begin{cases} bk - f & \text{if } \bar{y}^* > 0, \\ 0 & \text{if } \bar{y}^* = 0 \\ ak - f & \text{if } \bar{y}^* < 0. \end{cases}$$

It is easy to check that the following duality theorems hold:

$$\min_{x \in \mathcal{X}_k} P(x) = \min_{y^* \in \mathcal{Y}_s^*} P^d(y^*),$$

$$\max_{x \in \mathcal{X}_k} P(x) = \max_{y^* \in \mathcal{Y}_s^*} P^d(y^*).$$

The graphs of $P(x)$ and $P^d(y^*)$ are shown in Fig. 1..

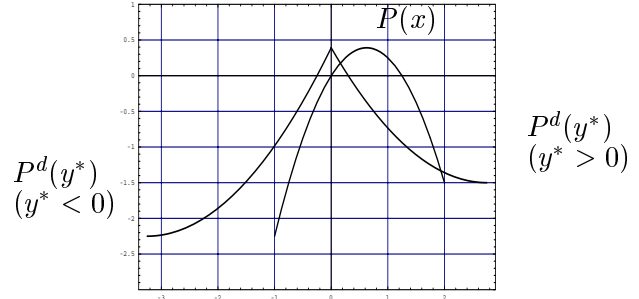


Figure 1. Bi-duality in constrained nonconvex optimization

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