Pergamon

Mechanics Research Communications, Vol. 23, No. 1, pp. 11-17, 1996

0093-6413/96 $12.00 + .00

0093-6413(95)00071-2

Nonlinear Elastic Beam Theory with Application in Contact Problems and Variational Approaches[1]

David Yang Gao
Department of Mathematics
Virginia Polytechnic Institute & State University
Blacksburg, VA 24061, USA. E-mail: gao@math.vt.edu

(Received 1 May 1995; accepted for print 19 October 1995)

1. Introduction

In the traditional nonlinear elastic beam theory, the stress in lateral direction is neglected and the governing equations can be considered as the one-dimensional von Karman model (see [1]):

$$\sigma_{x,x} = 0, \quad EIw_{,xxxx} - \sigma_x w_{,xx} - q = 0.$$

From the first equation we know that σ_x is a constant, so this beam model is actually a linear ordinary differential equation. If the beam is quite thick, the deformation in lateral direction can not be ignored. By considering the stress in lateral direction, a nonlinear beam theory is developed in this paper for large displacement and small strain elastic beam theory. Application to the unilateral problem with obstacle is illustrated and a nonlinear complementarity problem is proposed. We proved that this nonlinear complementarity problem is equivalent to a variational inequality and a primal variational approach.

2. Nonlinear Elastic Beam Model

We begin by describing the elastic beam model under large deflections. As shown in Figure 1, an elastic beam whose cross section in the x, y plane is a rectangle domain: $\Omega = \{(x, y) \in \mathbf{R}^2 | 0 \leq x \leq L, \ -h \leq y \leq h\}$, is subjected to both vertical and horizontal external loading systems. In this paper, we employ the fundamental hypothesis in the beam theory, i.e. cross sections which are perpendicular to the centroid locus before bending remain plane and perpendicular to the deformed locus. (An extended Timoshenko beam model has been developed in [2] without using this hypothesis.)

[1]This work was supported in part by National Science Foundation Grant DMS-9400565

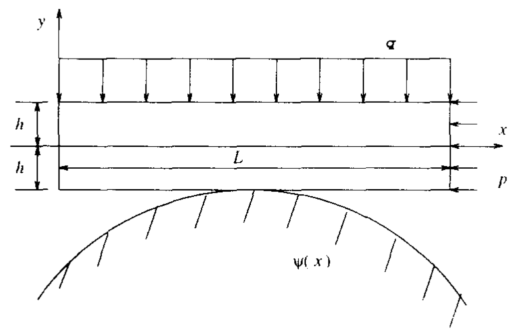

Figure 1. Contact problem of nonlinear elastic beam model.

By $w(x)$, we denote the transverse displacement of the elastic axis, which is the function of x only. The bending angle, measured in the mathematically positive direction, is $\theta = \tan^{-1}(\partial w/\partial x)$. Since the beam is subjected the x-axis directional force, we have to introduce an independent horizontal displacement $u(x)$ of the middle axis $y = 0$. So the total horizontal displacement ξ and lateral direction displacement η of the beam at the material point $(x, y) \in \Omega$ can be described as following:

$$\mathbf{u} = \begin{bmatrix} \xi(x,y) \\ \eta(x,y) \end{bmatrix} = \begin{bmatrix} u(x) - y\theta(x) \\ w(x) \end{bmatrix}. \tag{1}$$

By the definition of the Green-St Venant strain tensor in finite deformation theory, $\mathbf{E} = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^t + (\nabla\mathbf{u})^t(\nabla\mathbf{u})]$, the strain in this two dimensional elastic beam should be

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_{yy} \end{bmatrix} = \begin{bmatrix} u_{,x} - y\theta_{,x} + \frac{1}{2}(u_{,x} - y\theta_{,x})^2 + \frac{1}{2}w_{,x}^2 & \frac{1}{2}(w_{,x} - \theta) - u_{,x}\theta \\ \frac{1}{2}(w_{,x} - \theta) - u_{,x}\theta & \frac{1}{2}\theta^2 \end{bmatrix} \tag{2}$$

In this paper, we use notation $(\ )_{,x} := \frac{\partial}{\partial x}(\ )$, $(\ )_{,xy} := \frac{\partial^2}{\partial x \partial y}(\ )$, ..., for convenience. For the moderate large deflections of beam problems, we may assume that $h/L \sim w(x) \in O(1)$, $u \sim w_x \in O(\epsilon)$. Thus $u_{,x} \sim w_{,xx} \in O(\epsilon^2)$, where notation $\sim$ stands for "same order of magnitude". By Taylor expansion, we have $\theta = \tan^{-1} w_{,x} = w_{,x} + O(\epsilon^3)$. Neglecting terms higher than $O(\epsilon^2)$, and by using the engineering strain notations: $\epsilon_x = \epsilon_{xx}, \epsilon_y = \epsilon_{yy}, \gamma = 2\epsilon_{xy}$, we then have

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma \end{bmatrix} = \begin{bmatrix} u_{,x} - yw_{,xx} + \frac{1}{2}w_{,x}^2 \\ \frac{1}{2}w_{,x}^2 \\ 0 \end{bmatrix} \tag{3}$$

For large displacement but small strain plane deformation problem, the elastic constitutive relation can be given as

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y), \quad \sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x). \tag{4}$$

Suppose the beam is subjected lateral distributive loading system $q(x)$, and horizontal loading systems with magnitude p at the end $x = L$ in the negative x-direction. The potential energy of the elastic beam can then be written as

$$P(u,w) = \int_\Omega \frac{E}{2(1-\nu^2)}(\epsilon_x^2(x,y) + \epsilon_y^2(x) + 2\nu\epsilon_x(x,y)\epsilon_y(x))\mathrm{d}\Omega - \int_0^L q(x)w\mathrm{d}x \pm pu(x)|_{x=0,L}. \tag{5}$$

The first variation of P with respect to the displacements u, w is:

$$\begin{aligned}\delta P(u,w;\delta u,\delta w) &= \int_{-h}^{h} \sigma_x \delta u|_{x=0,L} dy - \int_\Omega \sigma_{x,x}\delta u \mathrm{d}\Omega \\ &- \int_{-h}^{h} y\sigma_x \delta w_{,x}|_{x=0,L} dy + \int_{-h}^{h} y\sigma_{x,x}\delta w|_{x=0,L} dy - \int_\Omega y\sigma_{x,xx}\delta w \mathrm{d}\Omega \\ &- \int_\Omega [(\sigma_{x,x} + \sigma_{y,x})w_{,x} + (\sigma_x + \sigma_y)w_{,xx}]\delta w \mathrm{d}\Omega \\ &+ \int_{-h}^{h} (\sigma_x + \sigma_y)w_{,x}|_{x=0,L}\delta w dy - \int_0^L q(x)\delta w \mathrm{d}x \pm p\delta u(x)|_{x=0,L}.\end{aligned}$$

Since u, w are functions of x only, this variational equality $\delta P = 0$ gives the Euler-Lagrange equation in the domain:

$$\int_{-h}^{h} \sigma_{x,x} dy = 0 \ \ \forall x \in [0, L], \tag{6}$$

$$\int_{-h}^{h} [y\sigma_{x,xx} + (\sigma_x + \sigma_y)_{,x} w_{,x} + (\sigma_x + \sigma_y)w_{,xx}]dy + q = 0 \tag{7}$$

Whereas, the boundary variational terms

$$\int_{-h}^{h} \sigma_x|_{x=0,L}\delta u dy = \mp p\delta u(x)|_{x=0,L}, \tag{8}$$

$$\int_{-h}^{h} y\sigma_x|_{x=0,L}\delta w_x dy = 0, \tag{9}$$

$$\int_{-h}^{h} (y\sigma_{x,x} + (\sigma_x + \sigma_y)w_x)|_{x=0,L}\delta w dy = 0. \tag{10}$$

will give the corresponding boundary conditions.

Substituting the constitutive relations (4) into (6,7), if E is a constant, we may obtained:

$$u_{,xx} + (1+\nu)w_{,x}w_{,xx} = 0 \ \ \forall x \in [0, L], \tag{11}$$

$$EIw_{,xxxx} - 2hE[(1+\nu)(2w_{,x}^2 + u_{,x})w_{,xx} + \nu w_{,x}u_{,xx}] - f(x) = 0 \ \ \forall x \in [0, L] \tag{12}$$

where $I = 2h^3/3$, $f(x) = (1 - \nu^2)q(x)$. These are coupled nonlinear fourth-order partial differential equations. Integrating (11), we have

$$u_{,x} = -\frac{1}{2}(1+\nu)w_{,x}^2 - \frac{\lambda}{2h(1+\nu)},$$

where λ is a integral constant, which depends on x-directional external forces on the boundary $x = 0, L$. Substituting this equation together with (11) into (12), we obtain

$$EIw_{,xxxx} - E\alpha w_{,x}^2 w_{,xx} + \lambda w_{,xx} - f(x) = 0 \quad \forall x \in [0, L], \tag{13}$$

where $\alpha = 3h(1 - \nu^2) > 0$ is a positive constant. This is the nonlinear beam equation we obtained in this paper.

Remark. If we ignore the lateral stress σ_y, then the Euler-Lagrange equations (6,7) of the variational problem $\delta P = 0$ should be

$$u_{,xx} + w_{,x}w_{,xx} = 0 \quad \forall x \in [0, L], \tag{14}$$

$$EIw_{,xxxx} - 2hE(u_{,x} + \frac{1}{2}w_{,x}^2)w_{,xx} - f(x) = 0 \quad \forall x \in [0, L] \tag{15}$$

which is equivalent to the following linear equation:

$$EIw_{,xxxx} + \lambda w_{,xx} - f(x) = 0 \quad \forall x \in [0, L]. \tag{16}$$

3. Complementarity Problem and Variational Approaches

In this section we are going to study the contact problem and its variational approaches. Let us consider a large deformed beam which is supported by a rigid obstacle G. The shape of this obstacle is given by a prescribed strictly concave function $\psi(x)$. Let $\mathcal{V}$ be an admissible displacement space such that for any given $w \in \mathcal{V}$, the derivatives and integration on $w(x)$ are allowed. For example, $\mathcal{V}$ could be $C^2[0, L]$ or $\mathcal{H}^2[0, L]$. Since the boundary conditions in this paper are not important, so we can simply assume that the beam is fixed at both ends. Then the kinematically admissible space $\mathcal{V}_a$ can be defined as

$$\mathcal{V}_a := \{w \in \mathcal{V} \mid w = 0, \;\; w_{,x} = 0 \;\text{ at } x = 0, L\}. \tag{17}$$

So the contact problem for this nonlinear elastic beam theory can be given as:

Problem 1 *(CP) For the given external loading system $(\lambda, f(x))$ and the concave obstacle function $\psi(x)$, find $w(x) \in \mathcal{V}_a$ such that*

$$EIw_{,xxxx} - E\alpha w_{,x}^2 w_{,xx} + \lambda w_{,xx} - f(x) \geq 0 \quad \forall x \in [0, L], \tag{18}$$

$$w(x) - \psi(x) \geq 0 \quad \forall x \in [0, L] \tag{19}$$

$$(EIw_{,xxxx} - E\alpha w_{,x}^2 w_{,xx} + \lambda w_{,xx} - f(x))(w - \psi) = 0 \quad \forall x \in [0, L]. \tag{20}$$

This is a nonlinear complementarity problem ((CP) for short). For any given solution w of the Problem (CP), the contact region $Z \subset [0, L]$ is defined as

$$Z := \{x \in [0, L] \mid w(x) = \psi(x) \;\; \forall x \in [0, L]\}.$$

Since the the boundary ∂Z of this contact region is not known before the obstacle problem is solved, ∂Z is called a free boundary. The eqn (20) is called the complementarity condition. The analytical solution of this nonlinear complementarity problem is very difficult. So its variational approaches will be useful for numerical methods.

Let $\mathcal{K}$ be a closed convex subset of $\mathcal{V}_a$:

$$\mathcal{K} = \{w \in \mathcal{V}_a \mid \; w(x) \geq \psi(x) \;\; a.e. \text{ in } [0, L]\} \tag{21}$$

The total potential energy $J : \mathcal{K} \to \mathbf{R}$ then can be given as

$$J(w) = \int_0^L \frac{1}{2} EI(w_{,xx})^2 \mathrm{d}x + \int_0^L \frac{1}{12} E\alpha(w_{,x})^4 \mathrm{d}x - \int_0^L \frac{1}{2}\lambda w_{,x}^2 \mathrm{d}x - \int_0^L fw \mathrm{d}x \tag{22}$$

It is easy to prove that the directional-derivative of J at w in the direction v can be written as

$$\delta J(w; v) = \int_0^L (EIw_{,xxxx} - E\alpha w_{,x}^2 w_{,xx} + \lambda w_{,xx} - f(x))v \mathrm{d}x = \langle \delta J(w), v\rangle,$$

here

$$\delta J(w) = EIw_{,xxxx} - E\alpha w_{,x}^2 w_{,xx} + \lambda w_{,xx} - f(x)$$

is the Gâteaux derivative of $J(w)$, and $\langle *, * \rangle$ denotes the bilinear form. Then the variational inequality problem ((VI) for short) for this obstacle problem can be proposed as

Problem 2 *(VI) For the given external force system $(\lambda, f(x))$ find $w \in \mathcal{K}$ such that*

$$\delta J(w; v - w) \geq 0 \;\; \forall v \in \mathcal{K}, \tag{23}$$

Numerical method for solving variational inequality problem has been studied for many years (see, for example, [3]). Next section, we will prove that this variational inequality is equivalent to the complementarity problem. Furthermore, the variational problem ((VP) for short) associated with this obstacle problem can be given by

Problem 3 *(VP) For the given external forces $(\lambda, f(x))$, find $w \in \mathcal{K}$ such that*

$$J(w) \leq J(v) \;\; \forall v \in \mathcal{K}. \tag{24}$$

Based on this variational problem, the finite element methods can be used to solve this nonlinear contact problem.

4. Equivalence Principle

General speaking, for large deformation problem, the variational problem and variational inequality are not equivalent (see [4]). But for this nonlinear beam problem, we have the following theorem.

Theorem 1 *For any given external load system $(\lambda, f(x))$, the complementarity problem (CP) and the variational inequality (VI) are equivalent. If $\lambda \leq 0$, both problems are equivalent to the variational problem (VP).*

Proof. First, we prove that $(CP) \Rightarrow (VI)$. From the complementarity condition (20), we have

$$\langle EIw_{,xxxx} - E\alpha w_{,x}^2 w_{,xx} + \lambda w_{,xx} - f(x), w - \psi \rangle = 0 \ \ \forall x \in [0, L].$$

Due to the fact that $\delta J(w) = EIw_{,xxxx} - E\alpha w_{,x}^2 w_{,xx} + \lambda w_{,xx} - f \geq 0 \ \ \forall x \in [0, L]$ (18), and $w - \psi \geq 0, \ \ v - \psi \geq 0 \ \ \forall w, v \in \mathcal{K}$, we have

$$\langle \delta J(w), v - \psi \rangle - \langle \delta J(w), w - \psi \rangle \geq 0 \ \ \forall v \in \mathcal{K}.$$

This gives that

$$\langle \delta J(w), v - w \rangle = \delta J(w; v - w) \geq 0 \ \ \forall v \in \mathcal{K},$$

i.e. the solution of the complementarity problem also solves the variational inequality problem.

To prove (VI) $\Rightarrow$ (CP), we rewrite the variational inequality

$$\delta J(w; v - w) = \int_0^L (EIw_{,xxxx} - E\alpha w_{,x}^2 w_{,xx} + \lambda w_{,xx} - f)(v - w)\mathrm{d}x \geq 0 \ \ \forall v \in \mathcal{K}. \quad (25)$$

Let $\phi \in \mathcal{V}$ be a nonnegative function such that $\phi(x) \geq 0 \ \ \forall x \in [0, L]$; then $v = w + \phi \in \mathcal{K}$. So

$$\int_0^L (EIw_{,xxxx} - E\alpha w_{,x}^2 w_{,xx} + \lambda w_{,xx} - f)\phi \mathrm{d}x \geq 0 \ \ \forall \phi \geq 0$$

gives

$$EIw_{,xxxx} - E\alpha w_{,x}^2 w_{,xx} + \lambda w_{,xx} - f \geq 0.$$

Furthermore, in the variational inequality (25), if we take $v = \psi \in \mathcal{K}$ and $v = 2w - \psi = w - (\psi - w) \in \mathcal{K}$, then we have

$$\delta J(w; \psi - w) = \langle \delta J(w), \psi - w \rangle \geq 0, \quad \text{and} \quad \delta J(w; w - \psi) = \langle \delta J(w), w - \psi \rangle \geq 0$$

respectively. Since $\delta J(w) \geq 0$, $w - \psi \geq 0$, we must have $\delta J(w; w - \psi) = \langle \delta J(w), w - \psi \rangle = 0$. It gives the complementarity condition:

$$(EIw_{,xxxx} - E\alpha w_{,x}^2 w_{,xx} + \lambda w_{,xx} - f)(w - \psi) = 0.$$

Now we are going to prove that if $\lambda \leq 0$, $(VP) \Leftrightarrow (VI)$. We let $b(\epsilon, \epsilon) = \int_0^L \frac{2}{3} E\alpha\epsilon^2 \mathrm{d}x$ be a convex functional of ϵ. For any given e, ϵ, we have

$$\frac{1}{2}b(\epsilon, \epsilon) - \frac{1}{2}b(e, e) \geq \int_0^L \frac{2}{3} E\alpha e(\epsilon - e)\mathrm{d}x = b(e, \epsilon - e) \ \ \forall \epsilon, e.$$

For any given $v \in \mathcal{K}$, we let $\epsilon = \frac{1}{2}v_{,x}^2$, $e = \frac{1}{2}w_{,x}^2$ and $\phi = v - w$, so

$$\epsilon(v) = \frac{1}{2}w_{,x}^2 + w_{,x}\phi_{,x} + \frac{1}{2}\phi_{,x}^2.$$

Then we have

$$\frac{1}{2}b(\epsilon(v),\epsilon(v)) - \frac{1}{2}b(e(w),e(w)) \geq b(e(w), w_{,x}\phi_{,x} + \frac{1}{3}\int_0^L E\alpha w_{,x}^2\phi_{,x}^2 \mathrm{d}x \quad \forall v \in \mathcal{K}.$$

If $\lambda \leq 0$, the functional $g(v) = \int_0^L \frac{1}{2}(EIv_{,xx}^2 - \lambda v_{,x}^2 - fv)\mathrm{d}x$ is a convex functional of v on $\mathcal{K}$. So we have

$$g(v) - g(w) \geq \int_0^L [EIw_{,xx}(v-w)_{,xx} - \lambda w_{,x}(v-w)_{,x} - f(v-w)]\mathrm{d}x \quad \forall v \in \mathcal{K}.$$

Together we have

$$J(v) - J(w) \geq \delta J(w; v-w) + \frac{1}{3}\int_0^L E\alpha w_{,x}^2 (v_{,x} - w_{,x})^2 \mathrm{d}x \quad \forall v \in \mathcal{K}. \tag{26}$$

Since $\frac{1}{3}\int_0^L E\alpha w_{,x}^2(v_{,x} - w_{,x})^2\mathrm{d}x \quad \forall v \in \mathcal{K}$, so $\delta J(w; v-w) \geq 0 \quad \forall v \in \mathcal{K}$ gives $J(v) - J(w) \geq 0 \quad \forall v \in \mathcal{K}$. This shows that $(VI) \Rightarrow (VP)$.

On the other hand, if $J(w) \leq J(v) \quad \forall v \in \mathcal{K}$, then there exists a $\gamma > 0$ such that for any given $t \in [0, \gamma)$,

$$J(w) \leq J(w + t(v-w)) \quad \forall v \in \mathcal{K}.$$

i.e.

$$0 \leq \frac{d}{dt} J(w + t(v-w))|_{t=0^+} = \delta J(w; v-w).$$

This shows that $(VP) \Rightarrow (VI)$. ■

Conclusion Remark. In the elastic buckling analysis of beam problems, where $\lambda > 0$, the complementarity problem (CP) is a nonlinear unilateral bifurcation problem, which is still equivalent to the variational inequality (23). Based on this variational inequality approach, many numerical methods can be suggested for solving this very difficult unilateral buckling problem. If $f(x) = 0$, then (13) is a nonlinear eigenvalue problem. The variational approaches of this kind problem is studied in [5] for von Karman plates.

References

[1] K. Washizu, *Variational Methods in Elasticity and Plasticity*, Pergamon Press, New York, 1968.

[2] D.Y. Gao and D.L. Russell, A finite approach to optimal control of a "smart" beam, in *Int. Conf. Computational Methods in Struct. and Geotechnical Engineering,* Ed. by P.K.K. Lee, L.G. Tham and Y.K. Cheung, Hong Kong, Dec. 12-24, 1994, pp. 135-140.

[3] N. Kikuchi and J.T. Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM Philadelphia, 1988, 495pp.

[4] S.T. Yau and Y. Gao, Obstacle problem for von Karman equations, *Adv. in Appl. Math.*, **13**, 123-141, 1992.

[5] D.Y. Gao, Duality theory in nonlinear buckling analysis for von Karman equations, *Studies in Applied Mathematics,* 1994, 423-444.