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VIBRATIONS OF A NONLINEAR DYNAMIC BEAM BETWEEN TWO STOPS

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Abstract. This work extends the model developed by Gao (1996) for the vibrations of a nonlinear beam to the case when one of its ends is constrained to move between two reactive or rigid stops. Contact is modeled with the normal compliance condition for the deformable stops, and with the Signorini condition for the rigid stops. The existence of weak solutions to the problem with reactive stops is shown by using truncation and an abstract existence theorem involving pseudomonotone operators. The solution of the Signorini-type problem with rigid stops is obtained by passing to the limit when the normal compliance coefficient approaches infinity. This requires a continuity property for the beam operator similar to a continuity property for the wave operator that is a consequence of the so-called div-curl lemma of compensated compactness.

1. **Introduction.** This work studies a model for the vibrations of a nonlinear beam when one of its ends is constrained to move between two stops. The motivation for such a problem comes from noise and vibration studies in industrial settings. Beams can be found in many manufactured parts, and noise is often generated by the contact between the end of a beam and its support. Weak solutions to a model that describes the dynamic vibrations of an elastic or viscoelastic beam in contact with two stops were established in [11]. Contact with the two stops was modelled in two different ways. One approach is to assume that the stops are perfectly rigid and to describe the contact using the classical SIgnorini unilateral condition. A second approach is to assume that the stops allow for interpenetration of the contact surfaces and to describe this type of contact using a normal compliance condition.

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In a subsequent papers [3, 9], numerical simulations for a related problem were carried out. A model for the thermomechanical behaviour of a beam in contact with two stops which allowed for the general evolution of material damage was investigated in [2]. A related problem was studied in [6]. Underlying all this work is the assumption that the beam vibrations are described by a linear constitutive relation of Kelvin-Voigt type.

In [4, 5] Gao proposed alternative models for thin and moderately thick nonlinear beams with finite deformation that allowed for the study of buckling of the beam under lateral compressive traction. The usual approach to buckling, using the standard linear fourth order model, is to assume that buckling takes place once the first oscillatory mode of the beam is excited. In the Gao model such an assumption is not necessary, since the energy function is of the double-well type and buckling appears naturally when the system's zero steady state losses its stability. An analysis of the dynamic vibrations of this Gao nonlinear beam can be found in [1], where the existence and uniqueness of a local weak solution is established, a numerical finite elements algorithm is presented and numerical simulations are depicted. A contact problem for this nonlinear beam can be found in [12].

In the present paper we investigate a model for a Gao nonlinear viscoelastic beam that is fixed at one end and is constrained to move at the other between two stops. As in [11], we consider both stops that are rigid and stops that are reactive and model the contact using the Signorini unilateral condition and the normal compliance condition, as appropriate. We establish existence results for both problems and uniqueness for the reactive stops problem.

We can now describe the remaining sections of the paper. In section 2 we present the physical setting and the model. In section 3 we give variational formulations and statements of the main existence results for both rigid and reactive stops. In section 4 we introduce a truncated version of the reactive stop problem and prove existence of a weak solution using an abstract existence theory found in [8] and [10]. We use this result in section 5 to establish the existence of a local weak solution to the original reactive stop problem. Section 6 gives an energy balance result for this problem which is used to show that the local weak solution is actually global. In section 7 the existence result for the rigid stops problem is established. Finally, in section 8 we give our conclusions.

2. The Model.

A model for the vibrations of a nonlinear beam was constructed by Gao in [4, 5]. In this paper we will allow for the presence of viscosity and assume that the right end of the beam vibrates between two rigid or reactive stops while at the left end the beam is clamped. We assume that the contact with the stops is frictionless, leaving the case with friction for another study. We denote by $w = w(x,t)$ the displacement of the central axis of the beam at location x and time t, and scale the length of the beam so that $0 \le x \le 1$.

The position of the two stops are given as g_1 and g_2 where ($g_1 < 0 < g_2$). We also assume that a horizontal traction $p = p(t)$, which may cause buckling, is acting at $x = 1$, too. This physical setting is depicted in Fig. 1.

Following the model derived in [4, 5] (see also [1] for details), the equation of motion of the beam is

$$w_{tt} - \sigma_x = f, \tag{2.1}$$

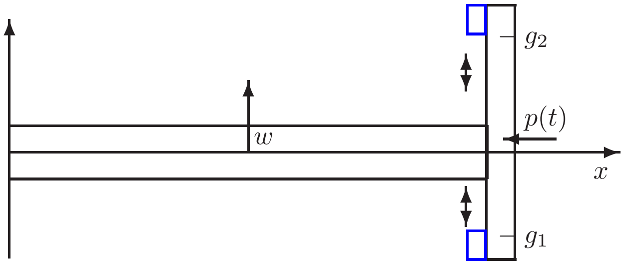

Figure 1. The beam and the two stops.

where the shear stress $\sigma = \sigma(x,t)$ is given by

$$\sigma = -kw_{xxx} - \gamma w_{txxx} + \frac{1}{3}aw_x^3 - \nu p w_x. \tag{2.2}$$

Here k is the scaled elasticity modulus, γ the viscosity coefficient, a the coefficient that allows for buckling of the beam, and ν is the scaled stiffness coefficient for lateral compression or tension. We suppose that when $p > 0$ the end $x = 1$ is being compressed and when $p < 0$ it is under tension.

The initial displacement and velocity of the beam are given by

$$w(x,0) = w_0(x), \qquad w_t(x,0) = v_0(x). \tag{2.3}$$

Since the beam is clamped at $x = 0$, we have that

$$w(0,t) = w_x(0,t) = 0. \tag{2.4}$$

At the right end, $x = 1$, we consider first the case when the beam may oscillate between two rigid stops. In this case the displacement $w(1,t)$ of the beam's end is restricted to move between the two stops; thus

$$g_1 \le w(1,t) \le g_2. \tag{2.5}$$

When $g_1 < w(1,t) < g_2$ the beam's end is not in contact with either of the two stops, therefore, $\sigma(1,t) = 0$. When $w(1,t) = g_1$ the beam's end is in contact with the lower stop and the reaction stress is such that $\sigma(1,t) \ge 0$, preventing the end from further movement down. When $w(1,t) = g_2$ the beam's end is in contact with the upper stop and the reaction stress satisfies $\sigma(1,t) \le 0$. These conditions can be expressed as

$$-\sigma(1,t) \in \partial I_{[g_1,g_2]}(w(1,t)), \tag{2.6}$$

where $\partial I_{[g_1,g_2]}$ refers to the subdifferential of the indicator function $I_{[g_1,g_2]}$ of the interval $[g_1, g_2]$. This is a Signorini-type unilateral condition.

In the second case we assume that the stops are reactive and behave as nonlinear springs and desribe that stress using the so-called normal compliance condition

$$-\sigma(1,t) = c_N \left((w(1,t) - g_2)_+ - (w(1,t) - g_1)_-\right). \tag{2.7}$$

Here c_N is the normal compliance stiffness coefficient, $(r)_+ = (|r| + r)/2$ is the positive part function, and $(r)_- = -(r - |r|)/2$ is the negative part function, so that $r = (r)_+ - (r)_-$. We note that $\sigma(1,t) < 0$ when $w(1,t) > g_2$; $\sigma(1,t) > 0$ when $w(1,t) < g_1$; and $\sigma(1,t) = 0$ when $g_1 \le w(1,t) \le g_2$.

Formally, (2.6) is obtained from(2.7) when $c_N \to \infty$.

The classical formulation of the problem of the *dynamic frictionless vibrations of the Gao beam between two rigid stops* may be stated as follows:

Problem P_{cl-S}. Find the displacement field $w = w(x,t)$ and the stress $\sigma = \sigma(x,t)$, for $x \in (0,1)$ and $t \in (0,T)$, such that

$$w_{tt} - \sigma_x = f, \tag{2.8}$$

$$\sigma = -kw_{xxx} - \gamma w_{txxx} + \tfrac{1}{3}a(w_x^3)_x - \nu p w_x, \tag{2.9}$$

$$w(0,t) = w_x(0,t) = 0, \tag{2.10}$$

$$kw_{xx}(1,t) + \gamma w_{txx}(1,t) = 0, \tag{2.11}$$

$$g_1 \le w(1,t) \le g_2, \tag{2.12}$$

$$-\sigma(1,t) \in \partial I_{[g_1,g_2]}(w(1,t)), \tag{2.13}$$

$$w(x,0) = w_0(x), \quad w_t(x,0) = v_0(x). \tag{2.14}$$

Here $f = f(x,t)$ is a given vertical force density which acts on the beam, and the condition (2.11) reflects the assumption that no moments act on the free end.

The classical formulation of the problem of the *dynamic frictionless vibrations of the Gao beam between two reactive stops* may be stated as follows:

Problem P_{cl-NC}. Find the displacement field $w = w(x,t)$ and the stress $\sigma = \sigma(x,t)$, for $x \in (0,1)$ and $t \in (0,T)$, such that (2.8)–(2.11), (2.7) and (2.14) hold.

We derive the weak or variational formulation of these two frictionless problems in the next section. Their analysis is given in Sections 5 and 7.

3. **Variational formulation and results.** In this section we present the variational formulations of problems P_{cl-NC} and P_{cl-S}, list the assumptions on the data, and state our existence results.

In what follows we will use standard notation for Sobolev spaces; in particular, we denote by $H^2(0,1)$ the space of (equivalence classes of) functions that are square integrable and have first and second square integrable distributional derivatives.

Let V be the closed subspace of $H^2(0,1)$ given by

$$V = \{z \in H^2(0,1) : z(0) = z_x(0) = 0\}.$$

We seek solutions w to the frictionless problems such that $w, w_t \in \mathcal{V} \equiv L^2(0,T:V)$ and $w_{tt} \in \mathcal{V}' \equiv L^2(0,T:V')$, where V' denotes the dual of V. We let $H = L^2(0,1)$ and denote by $(\cdot,\cdot)$ the usual inner product on H. Since $C([0,1])$ is dense in V and in H, we have that (V,H,V') is a Gelfand triple, and if we let $\mathcal{H} = L^2(0,T:H)$, then $(\mathcal{V},\mathcal{H},\mathcal{V}')$ is also a Gelfand triple. We denote the respective duality pairings by $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_{\mathcal{V}}$. On V we will use the norm $\|z\|_V^2 = (z_{xx}, z_{xx})$, which is equivalent to the usual $H^2(0,1)$ norm.

We first consider Problem P_{cl-NC}. Below, we use the prime to denote the (distributional) time derivative, and let $v = w_t = w'$. Applying (2.8) to a test function $u \in V$and using integration by parts gives, for $t \in (0,T)$,

$$\langle v'(t), u\rangle_V + \langle \sigma(t), u_x\rangle_V - \sigma(t)u|_0^1 = \langle f(t), u\rangle_V.$$

Then it follows from (2.7), and the fact that $u \in V$, that

$$-\sigma(t)u|_0^1 = -\sigma(1,t)u(1) = c_N\left((w(1,t)-g_2)_+ - (w(1,t)-g_1)_-\right)u(1).$$

Using (2.9), (2.10), (2.11) and the usual manipulations, we obtain the following variational formulation of the problem with reactive stops.

Problem P_{VNC}. Find the displacement field $w:[0,T]\to V$ and the velocity field $v:[0,T]\to V$, with $v'\in\mathcal{V}'$, such that for a.a. $t\in[0,T]$ and all $u\in V$,

$$\begin{aligned}\langle v'(t),u\rangle_V + k(w_{xx}(t),u_{xx}) + \gamma(v_{xx}(t),u_{xx}) + \frac{1}{3}a(w_x^3(t),u_x)\\ + c_N\left((w(1,t)-g_2)_+ - (w(1,t)-g_1)_-\right)u(1)\\ - \nu p(t)(w_x(t),u_x) = \langle f(t),u\rangle_V,\end{aligned} \tag{3.1}$$

$$w(t)=w_0+\int_0^t v(\tau)\,d\tau,\quad v(0)=v_0. \tag{3.2}$$

To obtain the variational formulation of the problem with rigid stops, we let K be the following closed and convex set in V where we seek $w(t)$,

$$K=\{z\in V:\ g_1\le z(1)\le g_2\}.$$

We now select a test function $u\in\mathcal{V}$ with $u(t)\in K$ and apply (2.8), together with (2.9), to $u-w$. Now, and this is the reason for applying to $u-w$, we observe that

$$-\sigma(1,t)(u(1,t)-w(1,t))\le 0,$$

which can be seen by checking the three different cases, since both $w(t)$ and $u(t)$ lie in K.

Using manipulations similar to those done in the first case we obtain the following variational formulation of the problem with rigid stops.

Problem P_{VS}. Find the displacement field $w:[0,T]\to K$ and the velocity field $v:[0,T]\to V$, with $v'\in\mathcal{V}'$, such that for each $u\in\mathcal{V}$ with $u'\in\mathcal{H}$, $u(t)\in K$ for a.a. $t\in[0,T]$, and $u(T)=w(T)$ we have

$$\begin{aligned}\int_0^T\langle v',u-w\rangle_V dt + k\int_0^T(w_{xx},u_{xx}-w_{xx})dt + \gamma\int_0^T(v_{xx},u_{xx}-w_{xx})dt\\ +\frac{1}{3}a\int_0^T(w_x^3,u_x-w_x)dt - \nu\int_0^T p(t)(w_x,u_x-w_x)dt\\ \ge \int_0^T(f,u-w)dt,\end{aligned} \tag{3.3}$$

together with (3.2). We note that in this case we have a variational inequality.

We make the following assumptions on the data:

$$w_0,\ v_0\in V, \tag{3.4}$$

$$p\in L^\infty(0,T),\quad |p|\le p^*, \tag{3.5}$$

$$f\in\mathcal{V}', \tag{3.6}$$

Here p^* is a positive constant.

The first existence and uniqueness result in this work is the following.

Theorem 3.1. *Assume that the conditions (3.4)–(3.6) hold. Then there exists a* $T^*>0$ *and a unique solution* (w,v) *of Problem* P_{VNC} *on the time interval* $[0,T^*)$*, such that,*

$$w\in L^\infty(0,T^*;V),\qquad v\in L^\infty(0,T^*;H)\cap\mathcal{V},\qquad v'\in\mathcal{V}'. \tag{3.7}$$

The proof of this result can be found in Section 5 and Section 6, based on the results for the truncated problems studied in Section 4. Moreover, using the energy estimate in Section 6, we show in Proposition 4 that, in fact, the solution exists on each time interval $[0, T]$, for $T < \infty$.

We conclude that Problem P_{cl-NC} has a unique global weak solution.

The second result deals with the rigid stops problem.

Theorem 3.2. *Assume that the conditions (3.4)–(3.6) hold and that additionally* $w_0 \in K \cap H^4(0,1)$ *and* $f \in \mathcal{H}$. *Then, for any* $T > 0$ *there exists a solution* (w, v) *of Problem* P_{VS} *on the time interval* $[0, T]$, *such that*

$$w \in L^\infty(0,T;V), \quad v \in L^\infty(0,T;H) \cap \mathcal{V}, \quad v' \in \mathcal{V}'$$
$$w(t) \in K \quad for \quad a.a. \quad t \in [0,T]. \tag{3.8}$$

The proof of this result can be found in Section 7, and uses the previous result for the problem with reactive stops. The solution here is obtained as the limit of the previous solutions as $c_N \to \infty$.

We conclude that Problem P_{cl-S} has a weak solution, but uniqueness of the solution remains unresolved.

4. **Truncated problems.**

The proof of Theorem 3.1 is based on the truncation of the term with w_x^3, which otherwise leads to mathematical difficulties. Once we show the existence of the weak solutions to the approximate problems, we can use a continuity argument in Section 5 to obtain the local (in time) solution of Problem P_{VNC}.

To deal mathematically with the cubic term in w_x, we introduce the truncation function Ψ_R,

$$\Psi_R(r) = \begin{cases} R & \text{for } R \le r, \\ r & \text{for } |r| \le R, \\ -R & \text{for } r \le -R, \end{cases} \tag{4.1}$$

where R is a large number. Then, we replace the term w_x^3 with $\Psi_R^2(w_x)w_x$. This gives us the following truncated variational problem.

Problem P_{VNCR}. Find the displacement and velocity fields $w, v : [0, T] \to V$ such that for a.a. $t \in [0, T]$ and all $u \in V$,

$$\begin{aligned} &\langle v'(t), u\rangle_V + k(w_{xx}(t), u_{xx}) + \gamma(v_{xx}(t), u_{xx}) \\ &+\tfrac{1}{3}a(\Psi_R^2(w_x(t))w_x(t), u_x) - \nu p(t)(w_x(t), u_x) \\ &+c_N((w(1,t) - g_2)_+ - (w(1,t) - g_1)_-, u) = (f(t), u), \end{aligned} \tag{4.2}$$

together with (3.2).

We have the following result for this problem.

Proposition 1. *Assume that (3.4)–(3.6) hold. Then, for each* $R > 0$ *and* $T > 0$ *there exists a unique solution* $(w, v) = (w_R, v_R)$ *to Problem* P_{VNCR} *on* $[0, T]$ *such that*

$$w \in C([0,T];V), \qquad v \in \mathcal{V}, \qquad v' \in \mathcal{V}'. \tag{4.3}$$

To prove the existence of a weak solution for Problem P_{VNCR}, we write it in an abstract form and use the results in [8, 10]. To that end, we define the operators

$B, K, K_R, J : \mathcal{V} \to \mathcal{V}'$, for $w, \phi \in \mathcal{V}$, by

$$\langle B(w), \phi \rangle_{\mathcal{V}} = \int_0^T \int_0^1 p(t) w_x \phi_x \, dxdt, \tag{4.4}$$

$$\langle K(w), \phi \rangle_{\mathcal{V}} = \int_0^T \int_0^1 w_{xx} \phi_{xx} \, dxdt, \tag{4.5}$$

$$\langle K_R(w), \phi \rangle_{\mathcal{V}} = \int_0^T \int_0^1 \Psi_R^2(w_x) w_x \phi_x \, dxdt, \tag{4.6}$$

$$\langle J\, w, \phi \rangle_{\mathcal{V}} = c_N \int_0^T ((w(1,t) - g_2)_+ - (w(1,t) - g_1)_-) \phi(1,t) \, dt, \tag{4.7}$$

The abstract formulation of (4.2) and (3.2) is as follows

Problem P_{AR}. Find a pair $(w, v) \in \mathcal{V} \times \mathcal{V}$ such that

$$v' + kK(w) + \gamma K(v) + \frac{1}{3} a K_R(w) - \nu B(w) + J(w) = f \quad \text{ in } \mathcal{V}', \tag{4.8}$$

together with (3.2).

We now rewrite this problem as a first order system. To do so we let $Y = V \times V$ and $\mathcal{Y} = \mathcal{V} \times \mathcal{V}$, and use the product norm $\|y\|_Y = \|\phi\|_V + \|\psi\|_V$, for $y = (\phi, \psi) \in Y$, and similarly for $\mathcal{Y}$. We define the operator $A : \mathcal{Y} \to \mathcal{Y}'$ by

$$A \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} -K(v) \\ kK(w) + \gamma K(v) + \frac{1}{3} a K_R(w) - \nu B(w) + J(w) \end{pmatrix}, \tag{4.9}$$

and $D : \mathcal{Y} \to \mathcal{Y}'$ by

$$D \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} K(w) \\ v \end{pmatrix}. \tag{4.10}$$

Let $F = (0, f)$ and $u_0 = (w_0, v_0)$, and let $\mathcal{Z} = \{u \in \mathcal{Y} : (Du)' \in \mathcal{Y}'\}$ with norm given by $\|z\|_{\mathcal{Z}} = \|z\|_{\mathcal{Y}} + \|(Dz)'\|_{\mathcal{Y}'}$. We may now rewrite Problem P_{AR} as follows.

Find $u = (w, v) \in \mathcal{Z}$ such that

$$(Du)' + Au = F, \qquad \text{in } \mathcal{Y}' \tag{4.11}$$

$$Du(0) = Du_0, \qquad \text{in } Y'. \tag{4.12}$$

Proof of Proposition 4.1. The first order system above is an implicit evolution problem of the type considered in [8, 10]. To use the existence results of these papers it suffices to show, that for all sufficiently large λ , the following three conditions hold true:

1. There exist constants C_0 and C_1, which depend on the data but that are independent of $u \in \mathcal{Y}$ such that $||Au||_{\mathcal{Y}'} \le C_0 + C_1 ||u||_{\mathcal{Y}}$ for all $u \in \mathcal{Y}$.
2. $\displaystyle\lim_{||u||_{\mathcal{Y}} \to \infty} \frac{\langle \lambda Du, u \rangle_{\mathcal{Y}} + \langle Au, u \rangle_{\mathcal{Y}}}{||u||_{\mathcal{Y}}} = \infty.$
3. $u \to (\lambda D + A)u$ is a pseudomonotone map from $\mathcal{Z}$ to $\mathcal{Z}'$.

We now derive the necessary estimates needed to establish these conditions. We set $u = (w, v) \in \mathcal{Y}$ below and denote by C, C_0 and C_1 generic positive constants whose values may change from line to line but which are independent of u.

Lemma 4.1. *The operator A satisfies condition 1 above.*

Proof. We consider in turn the various terms which appear in $\langle A(u), y\rangle_{\mathcal{Y}}$ where $y = (\phi, \psi) \in \mathcal{Y}$. First, we have that

$$|\langle K(v), \phi\rangle_{\mathcal{V}}| \leq \int_0^T \int_0^1 |v_{xx}||\phi_{xx}|\, dxdt \leq \|v\|_{\mathcal{V}}\|\phi\|_{\mathcal{V}} \leq \|u\|_{\mathcal{Y}}\|y\|_{\mathcal{Y}}.$$

Similarly, $|\langle K(w), \psi\rangle_{\mathcal{V}}| \leq \|u\|_{\mathcal{Y}}\|y\|_{\mathcal{Y}}$.

Next,

$$|\langle K_R(w), \psi\rangle_{\mathcal{V}}| \leq R^2 \int_0^T \int_0^1 |w_x||\psi_x|\, dxdt \leq R^2\|w\|_{\mathcal{V}}\|\psi\|_{\mathcal{V}} \leq R^2\|u\|_{\mathcal{Y}}\|y\|_{\mathcal{Y}}.$$

This is where the truncation is used. Next, using (3.5),

$$|\langle B(w), \psi\rangle_{\mathcal{V}}| \leq \int_0^T \int_0^1 |p(t)||w_x||\psi_x|\, dxdt \leq p^*\|w\|_{\mathcal{V}}\|\psi\|_{\mathcal{V}} \leq p^*\|u\|_{\mathcal{Y}}\|y\|_{\mathcal{Y}}.$$

Finally,

$$\begin{aligned} |\langle J(w), \psi\rangle_{\mathcal{V}}| &= c_N \int_0^T |[(w(1,t) - g_2)_+ - (g_1 - w(1,t))_+]|\psi(1,t)|\, dt \\ &\leq (C_0 + C_1\|w\|_{\mathcal{V}})\|\psi\|_{\mathcal{V}} \leq (C_0 + C_1\|u\|_{\mathcal{Y}})\|y\|_{\mathcal{Y}}. \end{aligned}$$

Here we have used a trace theorem for $\mathcal{V}$. Collecting these estimates shows that $|\langle A(u), y\rangle_{\mathcal{Y}}| \leq (C_0 + C_1\|u\|_{\mathcal{Y}})\|y\|_{\mathcal{Y}}$ which gives the result.

□

We now show coercivity.

Lemma 4.2. *The operator* $(\lambda D + A)$ *is coercive for all sufficiently large* λ*, i.e.,*

$$\lim_{||u||_{\mathcal{Y}}\to\infty} \frac{\langle \lambda Du, u\rangle_{\mathcal{Y}} + \langle Au, u\rangle_{\mathcal{Y}}}{||u||_{\mathcal{Y}}} = \infty. \tag{4.13}$$

Proof. We have,

$$\langle \lambda Du, u\rangle_{\mathcal{Y}} = \lambda\langle K(w), w\rangle_{\mathcal{V}} + \lambda\langle v, v\rangle_{\mathcal{V}} = \lambda\|w\|^2_{\mathcal{V}} + \lambda\|v\|^2_{\mathcal{H}}.$$

Next,

$$\begin{aligned} \langle Au, u\rangle_{\mathcal{Y}} &= \langle -K(v), w\rangle_{\mathcal{V}} + k\langle K(w), v\rangle_{\mathcal{V}} + \gamma\langle K(v), v\rangle_{\mathcal{V}} \\ &+ \frac{1}{3}a\langle K_R(w), v\rangle_{\mathcal{V}} - \nu\langle pB(w), v\rangle_{\mathcal{V}} + \langle J(w), v\rangle_{\mathcal{V}}. \end{aligned}$$

We deal with each term which appears on the right in turn. We use the Hölder inequality and the Cauchy inequality with ϵ in most of the estimates. First, we note that

$$\langle -K(v), w\rangle_{\mathcal{V}} \geq -\|v\|_{\mathcal{V}}\|w\|_{\mathcal{V}} \geq -\frac{\gamma}{8}\|v\|^2_{\mathcal{V}} - \frac{2}{\gamma}\|w\|^2_{\mathcal{V}}.$$

Similarly,

$$k\langle K(w), v\rangle_{\mathcal{V}} \geq -k\|v\|_{\mathcal{V}}\|w\|_{\mathcal{V}} \geq -\frac{\gamma}{8}\|v\|^2_{\mathcal{V}} - \frac{2k^2}{\gamma}\|w\|^2_{\mathcal{V}}.$$

We have,

$$\gamma\langle K(v), v\rangle_{\mathcal{V}} = \gamma\|v\|^2_{\mathcal{V}}.$$

Next,

$$\frac{1}{3}a\langle K_R(w), v\rangle_{\mathcal{V}} \geq -\frac{1}{3}aR^2\|w_x\|_{\mathcal{H}}\|v_x\|_{\mathcal{H}} \geq -\frac{\gamma}{8}\|v\|^2_{\mathcal{V}} - \frac{2a^2R^4}{9\gamma}\|w\|^2_{\mathcal{V}}.$$

Also,

$$-\nu\langle B(w),v\rangle_{\mathcal{V}}\geq -\nu p^*\|w_x\|_{\mathcal{H}}\|v_x\|_{\mathcal{H}}\geq -\frac{\gamma}{8}\|v\|^2_{\mathcal{V}}-\frac{2\nu^2p^{*2}}{\gamma}\|w\|^2_{\mathcal{V}}.$$

Finally, using a trace theorem in $\mathcal{V}$,

$$\langle J(w),v\rangle_{\mathcal{V}}\geq(-C_1-C_0\|w\|_{\mathcal{V}})\|v\|_{\mathcal{V}}\geq-\frac{\gamma}{8}\|v\|^2_{\mathcal{V}}-\frac{2C_0^2}{\gamma}\|w\|^2_{\mathcal{V}}-C.$$

Collecting these estimates and rearranging the constants yields

$$\langle\lambda Du,u\rangle_{\mathcal{Y}}+\langle Au,u\rangle_{\mathcal{Y}}\geq(\lambda-C)\|w\|^2_{\mathcal{V}}+\lambda\|v\|^2_{\mathcal{H}}+\frac{1}{4}\gamma\|v\|^2_{\mathcal{V}}-C.$$

Dividing this estimate by $||u||_{\mathcal{Y}}=||w||_{\mathcal{V}}+||v||_{\mathcal{V}}$ and letting $||u||_{\mathcal{Y}}\to\infty$ leads to (4.13), for each sufficiently large λ. □

We note that the presence of the viscosity term is essential for obtaining this result.

Finally , we prove that the operator $\lambda D+A$ is pseudomonotone from $\mathcal{Z}$ into $\mathcal{Z}'$.

Lemma 4.3. *The operator $\lambda D+A:\mathcal{Z}\to\mathcal{Z}'$ is pseudomonotone for all λ sufficiently large.*

Proof. We note that we may write $\lambda D+A=A_1+A_2$ where

$$A_1\begin{pmatrix} w\\ v\end{pmatrix}=\begin{pmatrix}\lambda K(w)-K(v)\\ kK(w)+\gamma K(v)\end{pmatrix}, \tag{4.14}$$

and

$$A_2\begin{pmatrix} w\\ v\end{pmatrix}=\begin{pmatrix}0\\ \frac{1}{3}aK_R(w)-\nu B(w)+J(w)\end{pmatrix}. \tag{4.15}$$

It is easy to check that A_1 is, in fact, monotone, for all λ sufficiently large. . Since the sum of a monotone operator and a completely continuous operator is pseudomonotone ([14]), it suffices to show that A_2 is completely continuous. This, in turn, follows from examining the operators K_R, B and J. To this end, let $\{u_m\}=\{(w_m,v_m)\}$ be a sequence which converges weakly to $u=(w,v)$ in $\mathcal{Z}$. Then, $w_m\to w$ and $v_m\to v$ weakly in $L^2(0,T;V)$ as $m\to\infty$, and also $w'_m\to w'$ and $v'_m\to v'$ weakly in $L^2(0,T;V')$. It follows from Corollary 4 of [13], that $\{w_m\}$ and $\{v_m\}$ are relatively compact in $L^2(0,T;H^1(0,1))$. By passing to subsequences we may assume that $\{w_m\}$ and $\{v_m\}$ converge strongly in $L^2(0,T;H^1(0,1))$ and pointwise a.a. to w and v, respectively. We have for $\phi\in\mathcal{V}$,

$$|\langle(B(w_m)-B(w),\phi\rangle_{\mathcal{V}}|\leq p^*\|w_{mx}-w_x\|_{L^2(0,T;H)}\|\phi_x\|_{L^2(0,T;H)}.$$

Since $\|\phi_x\|_{L^2(0,T;H)}\leq\|\phi\|_{\mathcal{V}}$, we obtain

$$\|(B(w_m)-B(w))\|_{\mathcal{V}'}\leq C\|w_{mx}-w_x\|_{L^2(0,T;H)}\to 0,\quad m\to\infty.$$

Next,

$$\begin{aligned}|\langle(K_R(w_m)-K_R(w),\phi\rangle_{\mathcal{V}}|&\leq R^2\|w_{mx}-w_x\|_{L^2(0,T;H)}\|\phi_x\|_{L^2(0,T;H)}\\ &+\|(\Psi_R^2(w_{mx})-\Psi_R^2(w_x))w_x\|_{L^2(0,T;H)}\|\phi_x\|_{L^2(0,T;H)}.\end{aligned}$$

Now since the truncation function Ψ is continuous and bounded it follows that $\|(\Psi_R^2(w_{mx})-\Psi_R^2(w_x))w_x\|_{L^2(0,T;H)}\to 0$ by the dominated convergence theorem. It follows that

$$\|(K_R(w_m)-K_R(w)\|_{\mathcal{V}'}\to 0,\quad m\to\infty.$$

Finally, using straightforward manipulations and a trace theorem yields

$$|\langle J(w_m) - J(w), \phi\rangle_{\mathcal{V}}| \leq C\|w_m - w\|_{L^2(0,T;H^1(0,1))}\|\phi\|_{L^2(0,T;H^1(0,1))}.$$

Then, dividing both sides by $\|\phi\|_{\mathcal{V}}$, we obtain

$$\|J(w_m) - J(w)\|_{\mathcal{V}'} \leq C\|w_m - w\|_{L^2(0,T;H^1(0,1))} \to 0$$

as $m \to \infty$. We conclude that the operaror $A_2 : \mathcal{Z} \to \mathcal{Z}'$ is completely continuous. □

Since the conditions (1)-(3) have now been established we have a solution $u = (w, v) \in \mathcal{Z}$. This implies that w, $w' = v \in \mathcal{V}$ and so $w \in C([0, T]; V)$. This completes the proof of the existence part in Proposition 1. Thus, for each R sufficiently large, there exists a solution (w_R, v_R) to Problem P_{VNCR}. The proof of the uniqueness of the solution is straightforward, and is omitted here since a very similar proof can be found in the following section concerning the solution of Problem P_{VNC}.

5. Proof of Theorem 3.1.

We prove Theorem 3.1 in two steps. First, we show that for a sufficiently large R there exists a time $0 < T^*$ such that the truncation is not active on $[0, T^*)$. Therefore, the solution (w_R, v_R) of the truncated problem is also a local solution of the problem without truncation. Then, we prove that the solution is unique.

Proposition 2. *Assume that $R > 0$ is fixed and sufficiently large so that $\|w_{0x}\|_{L^\infty(0,1)} < R$. Then there exists $0 < T^*$, which depends on the problem data, such that for $0 \leq t < T^*$, the solution w_R of problem P_{VNCR} satisfies*

$$\|w_{Rx}\|_{L^\infty((0,1)\times[0,T^*))} < R. \tag{5.1}$$

We conclude that $\Psi_R^2(w_{Rx}) = w_{Rx}^2 = w_x^2$ on the time interval $[0, T^*)$ and, therefore, (w_R, v_R) is a solution of Problem P_{VNC} on this time interval.

Proof. It follows from Proposition 1 that

$$w \in C([0, T]; V),$$

and so the mappings $w_x : [0, T] \to H^1(0, 1)$ and $w_{xx} : [0, T] \to L^2(0, 1)$ are continuous and bounded. Next, the Hölder inequality yields

$$|w_x(x, t)| \leq \int_0^1 |w_{xx}(r, t)|\, dr \leq \|w_{xx}(t)\|_{L^2(0,1)}. \tag{5.2}$$

Let $h(t) = \|w_{xx}(t)\|_{L^2(0,1)}$ then $h : [0, T] \to \mathbb{R}$ is continuous on a compact set, so it is bounded. Since $h(0) = \|w_{0xx}\|_{L^2(0,1)} < R$, there exist $0 < T^* \leq T$ such that $h(t) = \|w_{xx}(t)\|_{L^2(0,1)} < R$ for all $t \in [0, T^*)$. It follows that $|w_x(x, t)| \leq \|w_x(t)\|_{L^\infty(0,1)} < R$ for all $t \in [0, T*)$, that is (5.1) holds, and the truncation is inactive on this time interval. □

We now show that the local solution of the problem is unique. However, because of the buckling possiblity, we expect that for certain parameter regimes and appropriate initial conditions and forces the solution is sensitive to the data.

Proposition 3. *The solution (w, v) of Problem P_{VNC} on the time interval $[0, T^*)$ is unique.*

Proof. Let (w_1, v_1) and (w_2, v_2) be two solutions of Problem P_{VNC} and write $w = w_1 - w_2,\ v = v_1 - v_2$, and so $w_i = w_0 + \int_0^t v_i\, d\tau$, for $i = 1, 2$.

We use $v(t)$ as a test function in (3.1) for (w_1, v_1) and (w_2, v_2), and subtract the resulting expressions. Recalling that $H = L^2(0, 1)$, we may write it, for a.a. $t \in (0, T^*)$, as

$$\begin{aligned}\frac{d}{dt}\|v(t)\|_H^2 + k\frac{d}{dt}\|w_{xx}(t)\|_H^2 + 2\gamma\|v_{xx}(t)\|_H^2 - 2\nu p(t)(w_x(t), v_x(t))&\\ + c_N(-(w_2(1,t) - g_2)_+ + (g_1 - w_2(1,t))_+&\\ + (w_1(1,t) - g_2)_+ - (g_1 - w_1(1,t))_+)v(1,t)&\\ = \frac{2a}{3}((w_{2x}(t))^3 - (w_{1x}(t))^3, v(t)).&\end{aligned}$$

Integrating over $0 \le \tau \le t$ yields

$$\begin{aligned}\|v(t)\|_H^2 + k\|w_{xx}(t)\|_H^2 + 2\gamma\int_0^t \|v_{xx}(\tau)\|_H^2\, d\tau - 2\nu\int_0^t p(\tau)(w_x(\tau), v_x(\tau))\, d\tau&\\ + c_N\int_0^t (-(w_2(1,\tau) - g_2)_+ + (g_1 - w_2(1,\tau))_+&\\ + (w_1(1,\tau) - g_2)_+ - (g_1 - w_1(1,\tau))_+)v(1,\tau)\, d\tau&\\ = \frac{2a}{3}\int_0^t ((w_{2x}(\tau))^3 - (w_{1x}(\tau))^3, v(\tau))\, d\tau,&\end{aligned}$$

where we used the fact that $w(0) = v(0) = 0$. Next, we note that, for a.a. t,

$$\begin{aligned}|-(w_2(1,t) - g_2)_+ + (g_1 - w_2(1,t))_+ + (w_1(1,t) - g_2)_+ - (g_1 - w_1(1,t))_+|&\\ \le 2|w_2(1,t) - w_1(1,t)| \le 2\|w_x(t)\|_H.&\end{aligned}$$

Likewise, $|v(1,t)| \le \|v_x(t)\|_H$, for a.a. t. Since $\|w_{ix}(t)\|_{L^\infty(0,1)} \le R$ for $i = 1, 2$, we also have that

$$\|w_{2x}^3(\tau) - w_{1x}^3(\tau)\|_H \le 3R^2\|w_{2x}(\tau) - w_{1x}(\tau)\|_H \le C\|w_{xx}(\tau)\|_H.$$

Combining these results and using the fact that $|p| \le p^*$ by assumption (3.5) we have

$$\begin{aligned}\|v(t)\|_H^2 + k\|w_{xx}(t)\|_H^2 + 2\gamma\int_0^t \|v_{xx}(\tau)\|_H^2\, d\tau&\\ \le 2\nu p^*\int_0^t \|w_x(\tau)\|_H\|v_x(\tau)\|_H\, d\tau + C\int_0^t \|w_x(\tau)\|_H\|v_x(\tau)\|_H\, d\tau&\\ + C\int_0^t \|w_x(\tau)\|_H\|v_x(\tau)\|_H\, d\tau.&\end{aligned}$$

Using the Cauchy inequality with ϵ in the three terms on the right-hand side and the fact that $\|v_x(t)\|_H \le \|v_{xx}(t)\|_H$ and $\|w_x(t)\|_H \le \|w_{xx}(t)\|_H$, leads to

$$\|v(t)\|_H^2 + k\|w_{xx}(t)\|_H^2 + \gamma\int_0^t \|v_{xx}(\tau)\|_H^2\, d\tau \le C\int_0^t \|w_{xx}(\tau)\|_H^2\, d\tau.$$

Gronwall's inequality now gives $\|w_{xx}(t)\|_H^2 = 0$ for all $t \in [0, T^*)$. In view of the boundary conditions at $x = 0$, $w \equiv 0$, and this implies that $v \equiv 0$ and so the solution is unique. □

This concludes the proof of the existence and uniqueness assertions of Theorem 3.1. To establish the regularity assertions requires the energy balance result of the next section.

6. **Energy balance and global solution.** We derive an energy balance for the problem P_{VNC}. Then, we use it to show that the solution in Theorem 3.1 exists on each time interval $[0, T]$, for $T < \infty$.

First, we derive the energy balance equation. Let (w, v) be the local solution of P_{VNC} We use $v(t)$ as a test function in (3.1), and for any $T < T^*$ and for a.a. $t \in [0, T]$ we obtain

$$\begin{aligned}\langle v'(t), v(t)\rangle_V &+ k(w_{xx}(t), v_{xx}(t)) + \gamma(v_{xx}(t), v_{xx}(t)) + \frac{1}{3}a(w_x^3(t), v_x(t))\\ &+ c_N\left((w(1,t) - g_2)_+ + (w(1,t) - g_1)_-\right) v(1,t)\\ &- \nu p(t)(w_x(t), v_x(t)) = \langle f(t), v(t)\rangle_V.\end{aligned}$$

We may rewrite this as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v(t)\|_H^2 &+ \frac{k}{2}\frac{d}{dt}\|w_{xx}(t)\|_H^2 + \gamma\|v(t)\|_V^2 + \frac{a}{12}\frac{d}{dt}\|w_{xx}^2(t)\|_H^2\\ &+ \frac{c_N}{2}\frac{d}{dt}\left((w(1,t) - g_2)_+^2 - (w(1,t) - g_1)_-^2\right)\\ &- \nu p(t)(w_x(t), v_x(t)) = (f(t), v(t))_V.\end{aligned}$$

Integration over $0 \le \tau \le t < T^*$ yields

$$\begin{aligned}\frac{1}{2}\|v(t)\|_H^2 &+ \frac{k}{2}\|w_{xx}(t)\|_H^2 + \gamma\int_0^t \|v(\tau)\|_V^2\, d\tau + \frac{a}{12}\|w_x^2(t)\|_H^2\\ &+ \frac{c_N}{2}\left((w(1,t) - g_2)_+^2 + (w(1,t) - g_1)_-^2\right)\\ &= \frac{1}{2}\|v_0\|_H^2 + \frac{k}{2}\|w_{0xx}\|_H^2 + \frac{a}{12}\|w_{0xx}^2\|_H^2\\ &+ \frac{c_N}{2}\left((w_0(1) - g_2)_+^2 + (w_0(1) - g_1)_-^2\right)\\ &+ \nu\int_0^t p(\tau)(w_x(\tau), v_x(\tau))\, d\tau + \int_0^t (f(\tau), v(\tau))_V\, d\tau. \qquad (6.1)\end{aligned}$$

The *system energy* at time t, $E = E(t)$, is defined as

$$\begin{aligned}E(t) =& \frac{1}{2}\|v(t)\|_H^2 + \frac{k}{2}\|w_{xx}(t)\|_H^2 + \frac{a}{12}\|w_x^2(t)\|_H^2\\ &+ \frac{c_N}{2}\left((w(1,t) - g_2)_+^2 + (w(1,t) - g_1)_-^2\right).\end{aligned}$$

It follows that

$$\begin{aligned}E(t) = E(0) &- \gamma\int_0^t \|v(\tau)\|_V^2\, d\tau\\ &+ \nu\int_0^t p(\tau)(w_x(\tau), v_x(\tau))\, d\tau + \int_0^t (f(\tau), v(\tau))_V\, d\tau. \qquad (6.2)\end{aligned}$$

The first integral on the right-hand side is the viscous dissipation term, the second is the work done by the horizontal traction at $x = 1$, and the third integral is the work done by the force f.

The following estimate, for $0 \le t \le T < T^*$, is obtained in a straightforward manner from (6.2) using the Hölder and the Cauchy inequalities, with $E(0) = E_0$:

$$\begin{aligned} E(t) + \gamma \int_0^t \|v(\tau)\|_V^2 \, d\tau \\ \le E_0 + \frac{\gamma}{4} \int_0^t \|v_x(\tau)\|_H^2 \, d\tau + \frac{(\nu p^*)^2}{\gamma} \int_0^t \|w_x(\tau)\|_H^2 \, d\tau \\ + \frac{1}{\gamma} \int_0^t \|f(\tau)\|_{V'}^2 \, d\tau + \frac{\gamma}{4} \int_0^t \|v(\tau)\|_V^2 \, d\tau, \end{aligned} \tag{6.3}$$

Now, applying the Gronwall inequality, we obtain the following estimate

$$\begin{aligned} \|v(t)\|_H^2 + k\|w_{xx}(t)\|_H^2 + \frac{a}{6}\|w_x^2(t)\|_H^2 + \gamma \int_0^t \|v(\tau)\|_V^2 \, d\tau \\ + c_N \left((w(1,t) - g_2)_+^2 + (w(1,t) - g_1)_-^2 \right) \le C_E(T), \end{aligned} \tag{6.4}$$

for all $0 \le t \le T < T^*$, where the constant $C_E(T)$ may be chosen as

$$C_E(T) = 2(E_0 + \|f\|_{L^2(0,T;V')}) \exp(c^* T), \tag{6.5}$$

where $c^* = (\nu p^*)^2/\gamma$.

This estimate gives the regularity assertions of Theorem 3.1 and completes the proof of that theorem. It also allows us to establish a truncation level R which is large enough so that solutions to the truncated Problem P_{VNCR} and the untruncated problem P_{VNC} must coincide on the entire interval $[0, T]$. This is the content of the next result.

Proposition 4. *The solution (w, v) of Problem P_{VNC} exists on each time interval $[0, T]$, for $T < \infty$.*

Proof. Let T be fixed, and let $C_E(T)$ be the constant in (6.5) corresponding to this T, and choose R_0 and R to satisfy

$$R_0 = \sqrt{\frac{1}{k} C_E(T)} < R.$$

Next, we denote by w_R the solution of the truncated problem P_{VNCR}, for the chosen R, and let

$$A = \{0 \le t \le T : \|w_{Rx}(t)\|_{L^\infty(0,1)} \ge R\}.$$

If $A = \emptyset$, then $\|w_{Rx}(t)\|_{L^\infty(0,1)} < R$, and hence $\Psi_R^2(w_{Rx}) = w_{Rx}^2$, for $0 \le t \le T$. Therefore, the truncation is inactive on the time interval $[0, T]$ and $w_R = w$ is a global solution to P_{VNC}. If $A \ne \emptyset$, let $T^* = \inf A$. Since $w_R \in C([0, T], V)$ we have that $\|w_{Rx}(T^*)\|_{L^\infty(0,1)} \ge R$ and hence $T^* > 0$. On the other hand, for $0 \le t < T^*$, we have $\|w_{Rx}(t)\|_{L^\infty(0,1)} < R$ and so $w_R = w$ is a solution to Problem P_{VNC} for $0 \le t < T^*$. It follows from (6.5) that $C_E(T^*) \le C_E(T)$ and hence from (6.4) we have, for $0 \le t < T^*$, that

$$\|w_{xx}(t)\|_H = \|w_{Rxx}(t)\|_H \le R_0 < R.$$

Hence, $\|w_{Rx}(t)\|_{L^\infty(0,1)} \le R_0 < R$ and so again by continuity we have $\|w_{Rx}(T^*)\|_{L^\infty(0,1)} \le R_0 < R$, a contradiction. The conclusion now follows.

□

7. **Proof of Theorem 3.2.** To prove the theorem we pass to the limit $c_N \to \infty$ in a sequence of solutions to Problem P_{VNC} and thus obtain a solution to Problem P_{VS}. Specifically, we take $c_N = 1/\varepsilon, \varepsilon > 0$ in this section, and let $(w_\varepsilon, v_\epsilon)$ denote the corresponding solution of Problem P_{VNC}. Whenever the notation requires it, we assume that an appropriate measurable representative has been chosen. The proof requires a delicate convergence property of the solutions to the nonlinear beam equation of this paper that has already been established for solutions of the standard beam equation in [11]. Both of these results parallel a continuity property for the wave operator that is a consequence of the so-called div-curl lemma of compensated compactness (see [7]).

We begin with the following result, which is a counterpart to Lemma 4.1 of [11]. Note, in here and in what follows, no assumption is made about the behaviour of the individual terms that appear in $v'_\varepsilon + w_{\varepsilon xxxx} + \gamma v_{\epsilon xxxx}$. Also, throughout this section, we assume that $f \in \mathcal{H}$ and $w_0 \in K \cap H^4(0,1)$.

Lemma 7.1. *Let $(w_\varepsilon, v_\epsilon)$ be the solution of Problem P_{VNC}, with $\varepsilon > 0$. Then, there exists a constant M, independent of ε, such that*

$$\|v'_\varepsilon + w_{\varepsilon xxxx} + \gamma v_{\epsilon xxxx}\|_{\mathcal{H}} \le M.$$

Proof. Let $\phi \in C_0^\infty((0,1) \times (0,T))$. Using ϕ as the test function in (3.1) yields

$$\begin{aligned} -\int_0^T (v_\varepsilon(t), \phi'(t))_H \, dt + k \int_0^T (w_{\varepsilon xx}(t), \phi_{xx}(t))_H \, dt + \gamma \int_0^T (v_{\varepsilon xx}(t), \phi_{xx}(t))_H \, dt \\ = \frac{1}{3} a \int_0^T ((w^3_{\varepsilon x}(t))_x, \phi(t))_H \, dt - \nu \int_0^T p(t)(w_{\varepsilon xx}(t), \phi(t))_H \, dt \\ + \int_0^T (f(t), \phi(t))_H \, dt. \end{aligned}$$

Now, since $w_0 \in K$, it follows that the constant $C_E(T)$ that appears in (6.4) is independent of ε and hence both $w_{\varepsilon xx}$ and $(w^3_{\varepsilon x})_x$ are uniformly bounded in $L^\infty(0,T;H)$. Consequently, using the Hölder inequality, the right-hand side above is bounded by $M\|\phi\|_{\mathcal{H}}$, for some constant M independent of ε, and so we obtain

$$|\langle v'_\varepsilon + w_{\varepsilon xxxx} + \gamma v_{\varepsilon xxxx}, \phi\rangle_{\mathcal{H}}| \le M\|\phi\|_{\mathcal{H}},$$

which proves the lemma. □

Now, using (6.4) and Lemma 7.1, we can pass to a subsequence so that,

$$\begin{aligned} v_\varepsilon &\to v \quad \text{weakly in } \mathcal{V}, && (7.1) \\ v_\varepsilon &\to v \quad \text{weak} * \text{ in } L^\infty(0,T;H), && (7.2) \\ w_\varepsilon &\to w \quad \text{weak} * \text{ in } L^\infty(0,T;V), && (7.3) \\ v'_\varepsilon + w_{\varepsilon xxxx} + \gamma v_{\varepsilon xxxx} &\to v' + w_{xxxx} + \gamma v_{xxxx} \quad \text{weakly in } \mathcal{H}. && (7.4) \end{aligned}$$

Now, V embeds compactly in $C^1([0,1])$ which in turn embeds in H. Consequently, it follows from (7.2) and (7.3) and Corollary 4 of [13] that we may pass to a further subsequence and obtain

$$w_\varepsilon \to w \quad \text{strongly in } C\left([0,T]; C^1(0,1)\right). \tag{7.5}$$

Next, we observe that it also follows from (6.4) that

$$\left((w_\varepsilon(1,t) - g_2)_+^2 + (w_\varepsilon(1,t) - g_1)_-^2\right) \le C_E(T)\varepsilon,$$

and so, since (7.5) holds, $w(t) \in K$ for $0 \le t \le T$.

The above results are sufficient to pass to the lower order terms which appear in (3.1). However, to deal with the higher order terms we need the following lemma, which is established in [11] under the additional hypothesis that $w_0 = 0$. We note that the general case may be obtained by applying Lemma 4.3 of that paper to the shifted sequence $w_\varepsilon - w_0$. This is a remarkable result in that a convergence conclusion is obtained for nonlinear terms under hypotheses which only involve weak covergence.

Lemma 7.2. *Suppose* $(w_\varepsilon, v_\varepsilon)$ *satisfies* (7.1) − (7.5). *Then,*

$$\begin{aligned}\limsup_{\varepsilon\to 0} \int_0^T \int_0^1 &\left(v_\varepsilon^2 - w_{\varepsilon xx}^2 - \gamma v_{\varepsilon xx} u_{\varepsilon xx}\right) dxdt \\ &\le \int_0^T \int_0^1 \left(v^2 - w_{xx}^2 - \gamma v_{xx} w_{xx}\right) dxdt. \end{aligned} \tag{7.6}$$

We now are in position to pass to the limit in (3.1). For this purpose, if $\psi \in \mathcal{V}$, we let

$$P_\varepsilon(\psi) = \frac{1}{\varepsilon}\left((\psi(1,t) - g_2)_+ - (\psi(1,t) - g_1)_-\right).$$

Then, it is easy to check that the operator P_ε is monotone, i.e.,

$$(P_\varepsilon(\psi_1) - P_\varepsilon(\psi_2), \psi_1 - \psi_2)_H \ge 0.$$

Now, let $u \in \mathcal{V}$ be such that $u' \in \mathcal{H}$, $u(T) = w(T)$ and $u(t) \in K$ for $0 \le t \le T$, so that $P_\varepsilon(u(t)) = 0$. Then, in (3.1) we choose $u - w_\varepsilon$ as a test function, and obtain

$$\begin{aligned} &-\int_0^T (v_\varepsilon, u' - v_\varepsilon)_H \, dt + k\int_0^T (w_{\varepsilon xx}, u_{xx} - w_{\varepsilon xx})_H \, dt + (v_0, u(0) - w_0)_H \\ &\quad + \gamma \int_0^T (v_{\varepsilon xx}, u_{xx} - w_{\varepsilon xx})_H \, dt + \int_0^T \langle P_\varepsilon(w_\varepsilon), u - w_\varepsilon\rangle dt \\ &\quad + \frac{1}{3}a\int_0^T (w_{\varepsilon x}^3, u_x - w_{\varepsilon x})_H dt - \nu \int_0^T (p)(w_{\varepsilon x}, u_x - w_{\varepsilon x})_H dt \\ &\quad = \int_0^T (f, u - w_\varepsilon)_H \, dt. \end{aligned}$$

Now, since $P_\varepsilon(u) = 0$ and P_ε is monotone, we have that

$$\int_0^T \langle P_\varepsilon(w_\varepsilon), u - w_\varepsilon\rangle dt = \int_0^T \langle P_\varepsilon(w_\varepsilon) - P_\varepsilon(u), u - w_\varepsilon\rangle dt \le 0.$$

Therefore,

$$\begin{aligned} &-\int_0^T (v_\varepsilon, u' - v_\varepsilon)_H \, dt + k\int_0^T (w_{\varepsilon xx}, u_{xx} - w_{\varepsilon xx})_H \, dt + (v_0, u(0) - w_0)_H \\ &\quad + \gamma \int_0^T (v_{\varepsilon xx}, u_{xx} - w_{\varepsilon xx})_H \, dt \\ &\quad + \frac{1}{3}a\int_0^T (w_{\varepsilon x}^3, u_x - w_{\varepsilon x})_H dt - \nu \int_0^T (p)(w_{\varepsilon x}, u_x - w_{\varepsilon x})_H dt \\ &\quad \ge \int_0^T (f, u - w_\varepsilon)_H \, dt. \end{aligned} \tag{7.7}$$

Now, using Lemma 7.2 and (7.1)-(7.5) we can pass to the limit in (7.7) and obtain (3.3). This concludes the proof of Theorem 3.2.

We conclude that the Signorini problem Problem P_{cl-S} has a weak solution. However, the uniqueness of the solution remains an open problem.

8. **Conclusions.**

The problem of the vibrations of a nonlinear beam when one of its ends is constrained to move between two reactive or rigid stops has been studied.

The contact has been modeled with the normal compliance condition for the deformable stops, and with the Signorini condition for the rigid stops.

The existence of a unique local weak solution to the problem with reactive stops is shown by using truncation and results for pseudomonotone operators. An energy balance has been derived for the problem which allowed to show that the local weak solution is a global solution.

The solution of the Signorini-type problem with rigid stops is obtained by passing to the limit when the normal compliance coefficient approaches infinity. The uniqueness in this case remains an open problem.

It is of interest to use this result to construct an efficient and convergent numerical scheme for the model and perform numerical simulations. Since this beam may buckle, the behavior of the solutions is likely to be very complex. Moreover, using the Fourier Transform, the noise generated by the banging of the beam's end on the stops may be analyzed. The control of this noise may be of interest, and so the problem of the control of the vibrations of the beam will be studied in subsequent work.

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