History of manifolds

Long complex development, huge literature, hopefully illuminates:

• what are mathematical objects, and
• how do we come to understand them?

Objective is to improve mathematical practice.

1845 The term introduced (Grassman, Plücker)

higher dimensional linear space

1855 Riemann

• Philosophical, nonlinear, setting for Riemann surfaces
• organizing concept, “useful language” (Poincaré)

1900 Poincaré

• defined by example
• unsubstantiated claims of remarkable properties
• solid theory only in dimension 2.
• strange mix of techniques
• consolidation inhibited by desire for coherent global picture
• van der Waerden “battlefield of methods”
1930 definitions of smooth, homology manifolds

- gave up on immediately seeing unified picture, though development of PL probably inhibited by expectation there would be a unified picture.
- definitions justified by technical success
- descriptive theory of individual objects

1955 Thom, Smale, Milnor

- now expected $H^*$, top, pl, smooth all to be different; different families have very different techniques.
- precise relations top/pl/smooth worked out,
- focus shift to behavior of whole families of manifolds
- constructive techniques

1970 “Expert world view” Sullivan, Wall

- divide into high/low dimensions rather than $H^*/$top/pl/smooth
- high ($\geq 5$) dimensions have elaborate “characteristic behavior” with class a perturbation on this
- low dimensions are $\leq 3$ (reasonable theory) and 4 (mystery)
- community focus shifts to low dimensions

1980 view sharpened

- Freedman: topological 4-manifolds largely exhibit “high dimensional” behavior
- Donaldson: smooth 4-manifolds really strange (weak descriptive theory)
1995 view sharpened

- Homology manifolds exhibit high dimensional manifold behavior (Q, BFMW)
- Seiberg-Witten invariants of smooth 4-manifolds

2004 view sharpened

- Perelman geometrization
- Oszvath-Szabo Floer theory
- “big” topological 4-manifolds?

Next:

- structure theory for smooth 4-manifolds? (Poincaré conjecture, classification)
- are there two low dimensions (≤ 3 and smooth 4) or is there a single characteristic low-dimensional “behavior” as in high dimensions?
- new theory of smooth high-dimensional manifolds?

H*/top/pl behavior qualitatively simpler than smooth; see past this?