1. Declare the topic of your project by meeting with the instructor by Thursday March 23.

2. You are welcome to select a topic from the list below, or to pursue a different topic related to matrix theory that interests you. Topics connected to your own research are particularly encouraged. Please discuss the subject matter and scope with the instructor before embarking on such a project.

3. Final write-ups are due (by email) by 5:25pm on Saturday 6 May 2017 at the latest. No late submissions will be accepted on this assignment.

4. Type your report using LATEX (or your favorite word processor).

5. The report should be at least five pages, potentially including graphics.

6. Each student should complete an independent project, but you are welcome to discuss the topic with your fellow students, the instructor, your advisor, etc.

7. The project will count for 15% of your final grade in the course.

8. 80% of the project grade will be based on mathematical/scientific content. 20% will be based on the quality of the exposition. See the attached rubric for details.

9. Projects will generally be based on a careful reading of an article or section of a book.

10. Basic expectations. All projects should: (a) tackle a topic beyond the scope of the lectures; (b) refer in detail to at least one source in the literature (papers/books), and show an appreciation for the broader literature on the topic; (c) clearly describe the theoretical setting and assumptions; (d) provide illustrations (e.g., small examples, MATLAB plots, etc.) of the concept in action, ideally beyond those contained in the article/book itself.

A few project ideas follow. Some of these articles are lengthy works with great theoretical depth; digesting a solid piece will make for a fine project. Other papers are approachable survey articles; a complete project will dig deeper into the literature to follow points raised in these surveys. You will discuss the scope of your project when you meet with the instructor to declare your project intention.


Tensors are extensions of matrices for higher dimensional data. The most common tensors (beyond vectors and matrices) have three modes, i.e., $T \in \mathbb{C}^{m \times n \times p}$, with entries $t_{j,k,\ell} \in \mathbb{C}$. For example, each “slice” of $T$, say $T_{:,i,:} \in \mathbb{C}^{m \times n}$, could represent an image, and the collection of slices could represent frames of a video, or different planes of a 3d medical scan. This survey article provides a great introduction to tensors and some of their interesting complexity, e.g., how low-rank approximation of tensors differs from low-rank approximation of matrices.

Recent years have seen tremendous interest in “RandNLA,” the use of randomization in traditional numerical linear algebra algorithms to tackle very large scale problems. Often this field takes classical algorithms (such as subspace iteration for eigenvalue computation) and uses recent results from random matrix theory to derive convergence bounds that hold with high probability. This article includes deep and important results whose proofs are nontrivial. Working through a section of the article would make a fine project.


Quadratic eigenvalue problems arose in Problem Sets 1 and 2; they are naturally associated with second-order differential equations, and have many applications in damped mechanical systems. This important survey article will introduce you to the variety of spectral behavior possible for such problems, along with many applications where they arise.


How do eigenvalues change when a matrix is subjected to very small perturbations? We will answer this question in class for diagonalizable matrices. The analogous theory for Jordan blocks is substantially more intricate, but also rather beautiful (if you like that kind of thing). This article provides the best description of these results. Understand this article, and you will prove your mastery of the Jordan form in all its subtlety.


How do the eigenvectors of a Hermitian matrix change when the matrix is perturbed? This paper contains the landmark Davis–Kahan “sin Θ” theorems describing how invariant subspaces are perturbed. If you seek a deep theoretical investigation of angles between subspaces, this masterpiece is for you.

• Extended investigation of the numerical range (field of values).

The numerical range has many intriguing properties beyond the basic facts we can cover in our lectures. There are many projects lurking here. Start by reading Chapter 1 of R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991 (linked from the class website, with free access from VT). A few potential projects: (1) work through the proof of the convexity of the numerical range and investigate the geometric properties of its boundary; (2) investigate “inverse numerical range” problems: given a point \( z \in W(A) \), find a unit vector \( x \in \mathbb{C}^n \) such that \( x^* A x = z \); (3) investigate a few of the many generalizations of the numerical range.

• Toeplitz matrix theory.

Toeplitz matrices have constant values on each diagonal; this structure arises in a vast array of applications ranging from differential equations to signal processing. The spectral properties of such matrices are incredibly rich; the topic supports a wide variety of projects. For example, one could study the asymptotic distribution of eigenvalues of Hermitian Toeplitz matrices (the theorem of Grenander and Szegő), or non-Hermitian Toeplitz matrices (the theorem of Schmidt and Spitzer). The non-Hermitian case is particularly intriguing, as the eigenvalues of finite Toeplitz matrices tend to a limit that is generally different from the spectrum of the analogous infinite dimensional Toeplitz operator. Pseudospectra unravel this puzzle; see the paper by Reichel and Trefethen (1992) or the book by Böttcher and Grudsky (2005). The latter authors also study condition numbers of Toeplitz matrices.
Random matrices have a rich history, with intriguing spectral properties. (For an introduction, simply type `plot(eig(randn(1600)/40),'k.'); axis equal` and enjoy.) For an overview of recent results and some applications:


For a discussion of condition numbers of random matrices, see:


Triangular random matrices are much more ill-conditioned than dense ones:


Inverse eigenvalue problems.

Given a set of eigenvalues, can one construct a matrix of a specified form that has those eigenvalues? This inverse eigenvalue problem has attracted considerable attention over the years, and it is relevant, for example, in the design of structures that do not vibrate when stimulated at certain frequencies (e.g., by earthquakes). The case of Hermitian tridiagonal matrices is particularly clean, and is classically motivated by the problem of locating beads on a vibrating string. For a discussion involving real experimental data, see:


The Lanczos method, tridiagonal matrices, quadrature, model reduction, continued fractions.


The Lanczos process reduces a Hermitian matrix to tridiagonal form. As this algorithm proceeds, one learns a wealth of information about the matrix: eigenvalue estimates (obeying Cauchy interlacing), quadrature rules based on approximate spectral measures, reduced-order models that match moments; one even finds connections to continued fractions and orthogonal polynomials. Chapter 3 of the book by Liesen and Strakoš provides a masterful description of the connections between all these objects: one of the most complete and satisfying corners of matrix theory.


A key problem in the study of dynamical systems concerns the identification of parameter values for which a stable system goes unstable. Consider systems of the form $M\dot{x}(t) = A(p)x(t)$. What is the smallest value of $|p|$ for which the matrix pencil $(A(p), M)$ has a purely imaginary eigenvalue? This paper shows how this question can be answered elegantly in terms of Lyapunov equations, leading to an algorithm that requires the solution of such an equation at each iteration.


Spectral graph theory is a vital subject concerned with eigenvalues and eigenvectors of graph adjacency matrices. From such matrices one can learn a wealth of information about the graph, including, for example, how best to partition a graph into subgraphs. Spielman’s survey article will give you an introduction to this rich and fascinating field; pick an appealing thread and do a deeper investigation of the topic. Chung’s book is the classic reference with many more theoretical details.

Matrix theory can be used to characterize the proximity of two nodes in a graph (or “network”). This article describes one way (using the exponential of a matrix) that we shall discuss in class, as well as a variant using the resolvent. Use this paper as a jumping-off point to study “centrality measures” for network analysis.


In 2014, Virginia Tech famously beat Ohio State, who went on to be National Champions. If we beat the champions, don’t we have a case for being champions ourselves? (Certainly not that year…!) No doubt you have worked similar logic: If Team A beat Team B, and Team B beat Team C, does that not suggest that Team A is better than Team C? Are sports victories transitive? A simple nonnegative matrix model can be used to rank teams in a manner that incorporates head-to-head results. Study this model and its more sophisticated variants.


In class we will briefly discuss latent semantic indexing, an application of the singular value decomposition for text analysis. How can you detect the proximity of documents? How do you adjust an existing matrix factorization when new documents are entered into your database? This survey paper gives an introduction to these ideas.
MATH 5524: Matrix Theory Project
Grading Rubric

Name: ________________________________

An excellent evaluation earns full points; a satisfactory evaluation earns partial points; an unsatisfactory evaluation earns few or no points.

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GUIDELINES FOR EVALUATION

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<th>Section</th>
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<tr>
<td>Intro/Summary</td>
<td>Fluent description of the problem and summary of project</td>
<td>Some relevant details missing from problem statement, summary</td>
<td>Poor motivation; scope or content of project unclear</td>
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<tr>
<td>Literature review</td>
<td>Cites main paper(s) and puts in the context of broader literature</td>
<td>Cites main paper(s), but limited investigation beyond</td>
<td>Contribution of main paper not clearly explained</td>
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<tr>
<td>Theoretical exposition</td>
<td>Clear discussion of key theoretical findings, perhaps with a proof or two</td>
<td>Describes the key ideas of the subject but weak on a few details</td>
<td>The report only shows a superficial understanding of the subject</td>
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<td>Examples / experiments</td>
<td>Includes carefully thought-out experiments (by hand or computation)</td>
<td>A few examples/experiments are included, repeated from the paper</td>
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<td>Quality of presentation</td>
<td>Carefully written and revised document, good typesetting, illustrations</td>
<td>Document is fluent and clear, but room for improvement in text or plots</td>
<td>Sloppy text, incoherent presentation, poor graphics, typesetting</td>
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