1. [20 points]
   This problem addresses the $\xi = \xi(x)$ term that appears in the formula
   \[
   f(x) - p_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{j=0}^{n}(x - x_j)
   \]
given in Theorem 1.3 in the course notes, and Section 2.2.2 of Gautschi’s book.

   (a) Write down the linear interpolant $p_1(x)$ for the function $f(x) = x^3$ at the interpolation points $x_0 = 0$ and $x_1 = b$. Show that $\xi(x)$ takes the unique value $\xi(x) = (x + b)/3$.
   
   (b) Write down the linear interpolant $p_1(x)$ for the function $f(x) = 1/x$ at the interpolation points $x_0 = 1$ and $x_1 = 2$. Explicitly write down the function $\xi(x)$ for this case, and find the extreme values $\min_{1 \leq x \leq 2} \xi(x)$ and $\max_{1 \leq x \leq 2} \xi(x)$.

   [Süli and Mayers, Gautschi]

2. [20 points]
   Recall that for $A \in \mathbb{C}^{n \times n}$, the linear system $Ac = f$ has a unique solution for any $f$ provided $\text{Ker}(A) = \{0\}$, where $\text{Ker}(A)$ denotes the kernel (null space) of $A$.
   
   If the kernel of $A$ is larger, i.e., if there is a nonzero vector $z \in \text{Ker}(A)$, then there are two possibilities:
   
   - If $f \not\in \text{Ran}(A)$, then there is no solution $c$ to the linear system $Ac = f$.
   - If $f \in \text{Ran}(A)$, then there are infinitely many solutions to the linear system $Ac = f$. In particular, if $\tilde{c}$ satisfies $A\tilde{c} = f$, then any $c$ of the form $c = \tilde{c} + \gamma z$ is also a solution, where $\gamma$ is an arbitrary constant.
   
   With these facts in mind, please answer the following questions.

   (a) Suppose we wish to construct a polynomial $p_5 \in \mathbb{P}_5$ that interpolates a function $f \in \mathbb{C}^2[-1,1]$ in the following (somewhat unusual) manner: $p_5(-1) = f(-1)$; $p_5'(0) = f'(0)$; $p_5''(0) = f''(0)$; $p_5(1) = f(1)$; $p_5'(1) = f'(1)$. Write down a linear system to determine the coefficients $c_0, \ldots, c_5$ for $p$ in the monomial basis: $p_5(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5$.
   
   (b) What is the kernel of the matrix $A$ constructed in part (a)?
      (You may use the MATLAB command `null(A, 'r'`).)
      What does your answer imply about the existence and uniqueness of the interpolant $p_5$?
   
   (c) Consider the data: $f(-1) = -1$, $f'(-1) = 0$, $f(0) = 1$, $f''(0) = -2$, $f(1) = 3$, $f'(1) = 4$. Show that there are infinitely many choices for the polynomial $p_5$ that interpolate this data. Plot six of them. (Superimpose all on the same plot.)

3. [20 points]
   The Hermite interpolant $h_n \in \mathbb{P}_{2n+1}$ of $f \in C^1[a,b]$ at the points $\{x_j\}_{j=0}^{n}$ can be written in the form
   \[
   h_n(x) = \sum_{j=0}^{n} \left( A_j(x)f(x_j) + B_j(x)f'(x_j) \right),
   \]
where the functions $A_j$ and $B_j$ generalize the Lagrange basis functions:

\[ A_j(x) = \left(1 - 2\ell_j'(x_j)(x - x_j)\right)\ell_j^2(x) \]
\[ B_j(x) = (x - x_j)\ell_j^2(x), \]

with $\ell_j(x) = \prod_{k=0, k\neq j}^{n}(x - x_k)/(x_j - x_k)$.

(a) Verify that

\[ A_j(x_k) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}, \quad A_j'(x_k) = 0, \quad B_j(x_k) = 0, \quad B_j'(x_k) = \begin{cases} 1 & j = k \\ 0 & j \neq k. \end{cases} \]

(b) The above expression for the Hermite interpolating polynomial mimics the Lagrange form of the standard interpolating polynomial. Devise a scheme for constructing Hermite interpolants that generalizes the Newton form. What are your new Newton-like basis functions for $P_{2n+1}$?

4. [20 points]

The one-dimensional interpolation scheme studied in class can be adapted to higher dimensions. For example, suppose we are given a scalar-valued function $f(x, y)$, such as

\[ f(x, y) = e^x \sin y, \]

and wish to construct a function of the form

\[ p(x, y) = c_0 + c_1x + c_2y + c_3xy + c_4x^2 + c_5y^2 \]

that interpolates $f(x, y)$ at $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5)$.

(a) Set up a linear system $Ac = f$ to determine the coefficients $c_0, \ldots, c_5$.

(b) Write a MATLAB code to determine $c$ when $f(x, y) = e^x \sin y$ and the $(x_j, y_j)$ pairs take the values listed in the following table.

<table>
<thead>
<tr>
<th>$j$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_j$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$y_j$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Report your value for $c$.

(c) Plot your model function $p(x, y)$ over $x \in [-1, 3], y \in [-1, 3]$ using MATLAB’s `surf` command. Compare this plot to the similar plot for $f(x, y)$, which can be obtained in the following manner.

\[
\text{f = inline('exp(x).*sin(y)','x','y');}
\text{[xx,yy] = meshgrid(linspace(-1,3,25),linspace(-1,3,25));}
\text{zz = f(xx,yy);}
\text{figure(1), clf}
\text{surf(xx,yy,zz)}
\]

Please submit plots of both $p(x, y)$ and $f(x, y)$.

5. [20 points]

Suppose the complex-valued function $f(z)$ of the variable $z \in \mathbb{C}$ is analytic in a region $D$ of the complex plane whose boundary $C$ is a simple closed contour. Furthermore, suppose the interpolation points $x_0, \ldots, x_n$ $(n \geq 1)$ and the point $x$ all lie in $D$. 

(a) Let \( p_n \in \mathcal{P}_n \) denote the polynomial that interpolates \( f \) at \( x_0, \ldots, x_n \).
For any \( x \in D \), confirm the identity

\[
f(x) - p_n(x) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-x} \prod_{j=0}^n \frac{x-x_j}{z-x_j} \, dz
\]

by computing the integral on the right. (Hint: Consider the poles of the integrand, and use the Cauchy integral formula.)

For the rest of the problem, suppose that the real number \( x \) and the interpolation points \( x_0, \ldots, x_n \) all lie in the real interval \([a,b]\), and define, for constant \( K > 0 \),

\[
D = \{ z \in \mathbb{C} : |z-t| < K \text{ for some } t \in [a,b] \}.
\]

(b) Plot (or draw) the boundary \( C \) of \( D \) for \([a,b] = [-1,1]\) and \( K = 1 \).

(c) Show that the length of the contour \( C \) is \( 2(b-a) + 2\pi K \), and that the integral formula in (a) leads to the bound

\[
|f(x) - p_n(x)| < \frac{(b-a + \pi K)M}{\pi K} \left( \frac{b-a}{K} \right)^{n+1},
\]

where \( M \) is such that \( |f(z)| \leq M \) on \( C \).

(d) Deduce that if \( f \) is analytic on \( D \) for some \( K > |b-a| \), then the sequence \( \{p_n\} \) converges to \( f \) uniformly on \([a,b]\) as \( n \to \infty \).

(e) Show that the requirements for the conclusion in (d) are not satisfied by Runge’s function, \( f(x) = 1/(1 + x^2) \) over \([a,b] = [-5,5]\). For what values of \( \alpha \) are the conditions satisfied by this \( f \) over \([a,b] = [-\alpha,\alpha]\)?

[Süli and Mayers, Problem 6.11]

6. [20 points]

The standard Lagrange interpolation formula for the polynomial \( p_n \in \mathcal{P}_n \) that interpolates \( f \in C[a,b] \) at the distinct points \( \{x_j\} \),

\[
p_n(x) = \sum_{j=0}^n \ell_j(x) f(x_j), \quad \text{where} \quad \ell_j(x) = \prod_{k=0, k \neq j}^n \frac{x-x_k}{x_j-x_k},
\]

requires \( O(n^2) \) floating point operations to evaluate for each point \( x \). In this exercise, we construct an alternative Lagrange interpolation formula, known as the barycentric interpolant, that can evaluate more efficiently and also has superior numerical stability.

Let \( w(x) = \prod_{k=0}^n (x-x_k) \) and define the barycentric weight as

\[
\beta_j = \frac{1}{\prod_{k=0, k \neq j}^n (x_j-x_k)}, \quad j = 0, \ldots, n.
\]

(a) Show that the Lagrange form for \( p_n \) can be rewritten as

\[
p_n(x) = w(x) \sum_{j=0}^n \frac{\beta_j}{x-x_j} f(x_j).
\]

(b) Verify that

\[
1 = w(x) \sum_{j=0}^n \frac{\beta_j}{x-x_j}.
\]

(Hint: This follows from part (a) with a special choice of \( f \).)
(c) Dividing the result of part (a) by the result of part (b) yields the barycentric interpolation formula

\[ p_n(x) = \frac{\sum_{j=0}^{n} \beta_j \frac{f(x_j)}{x-x_j}}{\sum_{j=0}^{n} \beta_j \frac{1}{x-x_j}}. \]

Assuming the \( \beta_j \) values are already known, how many floating point operations are required to evaluate \( p_n(x) \) for some point \( x \)?

(d) Suppose \( [a, b] = [0, 1] \) and \( x_j = j/n \) for \( j = 0, \ldots, n \). Derive a simple formula for \( \beta_j \) in terms of \( j \) and \( n \). For which \( j \) values is \( \beta_j \) largest (in absolute value)? (These terms will be favored in the formula in part (c).)

[Berrut and Trefethen]

7. [20 points]

As mentioned in class, the Weierstrass Approximation Theorem states that for any \( f \in C[a, b] \) and any \( \varepsilon > 0 \), there exists some polynomial (of unspecified degree) such that \( \max_{x \in [a, b]} |f(x) - p(x)| < \varepsilon. \)

The most common proof of this fact is constructive: one can use for the approximating polynomial the Bernstein polynomial of appropriate degree. When \( [a, b] = [0, 1] \), the degree-\( n \) Bernstein polynomial is defined as

\[ B_n(x) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k}, \]

where \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \) can be obtained in MATLAB via the \texttt{nchoosek} command.

Remarkably, it turns out that for any \( f \in C[a, b] \), we have \( \max_{x \in [a, b]} |f(x) - B_n(x)| \to 0 \) as \( n \to \infty \).

In this exercise you shall explore the rate at which this convergence occurs.

(a) Confirm that \( B_n(x) \to f(x) \) for \( x \in [0, 1] \) and \( f(x) = \sin(3\pi x) \) by producing a MATLAB plot that compares \( f(x) \) to \( B_n(x) \) on \( x \in [0, 1] \) for \( n = 5, 10, 20 \). (Please label the plot clearly!)

(b) Describe how to modify the definition of \( B_n \) so as to work for a general interval \( [a, b] \neq [0, 1] \).

(c) Let \( f(x) = e^x \) and \( [a, b] = [-1, 1] \). Write MATLAB code to compute \( B_n(x) \) as well as the polynomial \( p_n(x) \) that interpolates \( f \) at the Chebyshev points

\[ x_k = \cos(k\pi/n), \quad k = 0, \ldots, n. \]

(You may use the monomial, Newton, or Lagrange basis.) Turn in a \texttt{semilogy} plot that compares \( \max_{x \in [-1, 1]} |f(x) - B_n(x)| \) with \( \max_{x \in [-1, 1]} |f(x) - p_n(x)| \) for \( n = 1, \ldots, 40 \). (For purposes of this problem, you may ignore any warnings issued by \texttt{nchoosek} for large \( n \).)

(d) Repeat part (c) with \( f(x) = x^2 - 1 \) and \( [a, b] = [-1, 1] \).