The exam will focus on material covered since the midterm, but the key concepts from the first half of the course are fair game:

E.g., best approximation, symmetry of linear operators, eigenvalues and eigenfunctions, etc.

Key ideas from second half of the course:

* Spectral method for time-dependent problems
  
e.g., \( U_t = -Lu + f \), \( U_{tt} = -Lu + f \)

1. Write solution as an eigenfunction expansion:
   \[
   u(x, t) = \sum_{j=1}^{\infty} a_j(t) \psi_j(x).
   \]

2. Substitute this form into the PDE
   
   e.g., \( u_t = -Lu + f \) \( \Rightarrow \) \[ \sum_{j=1}^{\infty} a_j'(t) \psi_j(x) = -\sum_{j=1}^{\infty} \lambda_j a_j(t) \psi_j(x) + f(x, t) \]
   
   \[ \Rightarrow \sum_{j=1}^{\infty} a_j'(t) \psi_j(x) = \sum_{j=1}^{\infty} (-\lambda_j a_j(t)) \psi_j(x) + f(x, t). \]

3. Take the inner product with \( \psi_k \) and use orthogonality of eigenfunctions:
   
   \[ \sum_{j=1}^{\infty} a_j'(t) (\psi_j, \psi_k) = \sum_{j=1}^{\infty} (-\lambda_j a_j(t)) (\psi_j, \psi_k) + (f(t), \psi_k) \]
   
   = 0 if \( j \neq k \)
   
   = 0 if \( j = k \)
\[
\Rightarrow \quad a_k'(t) = -\lambda_k a_k(t) + \frac{(f(t_1), \psi_k)}{(\psi_k, \psi_k)} = c_k(t)
\]

\[
\Rightarrow \quad a_k'(t) = -\lambda_k a_k(t) + c_k(t)
\]

ODE for each coefficient \( a_k(t) \), \( k = 1, 2, \ldots \)

**Initial Condition Comes from the Initial Condition**

For the PDE:

\[
\lambda_k(0) = \frac{(U_0, \psi_k)}{(\psi_k, \psi_k)}
\]

4. Solve the scalar ODEs:

\[
e^{\lambda_k t} a_k(0) + \int_0^t e^{\lambda_k(s-t)} c_k(s) \, ds
\]

5. Assemble the solution:

\[
U(x,t) = \sum_{j=1}^{\infty} a_j(t) \psi_j(x)
\]

Then we might also add in a preliminary step to handle inhomogeneous boundary conditions.

6. Handle inhomogeneous boundary conditions.

Write

\[
U(x,t) = V(x,t) + W(x,t)
\]

where \( W(x,t) \) handles the boundary condition and \( V(x,t) \) satisfies a related PDE with homogeneous boundary conditions.

Note: Adjust \( U_0(x) \), \( f(x,t) \) to compensate for \( W(x,t) \).
Finite Element Methods for Time Dependent Problems

1. Derive a weak form of the PDE
   \[ u_t = u_{xx} + f \Rightarrow \frac{\partial}{\partial t} \langle u, v \rangle = -a(u, v) + (f, v) \quad \forall v \in V \]
   \[ uu_{tt} = u_{xx} + f \Rightarrow \frac{\partial^2}{\partial t^2} \langle u, v \rangle = -a(u, v) + (f, v) \quad \forall v \in V \]
   WHERE V is a suitable space of test functions
   (RECALL ESSENTIAL VS NATURAL BOUNDARY CONDITIONS)

2. Galerkin Approximation: Restrict V to a subspace
   \[ V_N = \text{Span} \{ \phi_1, \ldots, \phi_N \} \]
   e.g., for \( u_t = u_{xx} + f \), seek
   \[ u_N = \sum_{j=1}^{N} \phi_j(t) \phi_j(x) \]
   such that
   \[ \frac{\partial}{\partial t} \langle u_N, v \rangle = -a(u_N, v) + (f, v) \quad \forall v \in V_N \]

3. Derive a linear algebra problem:
   Imposing weak form on \( \phi_1, \ldots, \phi_N \) to get, e.g.,
   \[
   \begin{bmatrix}
   \langle \phi_1, \phi_k \rangle \\
   \vdots \\
   \langle \phi_N, \phi_k \rangle \\
   \end{bmatrix}
   \begin{bmatrix}
   \phi_1(t) \\
   \vdots \\
   \phi_N(t) \\
   \end{bmatrix}
   = -\begin{bmatrix}
   a(\phi_1, \phi_1) \\
   \vdots \\
   a(\phi_N, \phi_N) \\
   \end{bmatrix}
   \begin{bmatrix}
   \phi_1(t) \\
   \vdots \\
   \phi_N(t) \\
   \end{bmatrix}
   + \begin{bmatrix}
   \langle f, \phi_1 \rangle \\
   \vdots \\
   \langle f, \phi_N \rangle \\
   \end{bmatrix}
   \]
   Initial data \( a(0) \)
   comes from the initial condition, \( u_0 \)
   \[ M \phi_1(t) = -K \phi_1(t) + f(t) \]

4a. Solve the linear algebra problem exactly (in time)
   \[ a(t) = e^{-M^{-1}Kt} a(0) + \int_0^t e^{-M^{-1}K(s-t)} f(s) \, ds \]

   Matrix exponential: know basic properties, e.g.,
   \[ e^{tA} = I + tA + \frac{t^2A^2}{2} + \frac{t^3A^3}{3!} + \ldots \]
Recall $e^{\lambda t} \to 0$ as $t \to 0$ if and only if $\Re(\lambda) < 0$ for all eigenvalues $\lambda$ of $A$.

4.5 Solve the linear algebra problem approximately for the generic problem $y'(t) = Ay(t)$ (in time).

- **Forward Euler:** $y_{k+1} = y_k + \Delta t Ay_k$
- **Backward Euler:** $y_{k+1} = y_k + \Delta t Ay_{k+1}$

\[ \Rightarrow \quad y_{k+1} = (I - \Delta t A)^{-1} y_k \]

**Understand stability considerations for these methods:** E.g., for the heat equation, backward Euler is stable for all $\Delta t > 0$, but forward Euler is more restrictive: double $N$, quarter $\Delta t$.

This is the "CFL condition."

5. *Uncertainty quantification*

If this appears on the test, it will be at the conceptual level — no detailed calculations.

- **Frequentist perspective**
  - Least squares gives an unbiased estimator
  - We can also get an unbiased estimator for the variance of the noise

- **Bayesian perspective**
  - We get a distribution for the QoIs informed by observations.