Lecture 40: Bayesian Approach to Parameter Estimation

Having seen the frequentist approach to parameter estimation, we now consider basic Bayesian techniques; see Smith, Section 4.8 and Chapter 8.

Bayes Theorem

Let "A" and "B" denote two events

\[ P(A) = \text{Probability Event A occurs } \in [0,1] \]
\[ P(B) = \text{Probability Event B occurs } \in [0,1] \]
\[ P(A|B) = \text{Probability A occurs, given that B occurs} \]
\[ P(B|A) = \text{Probability B occurs, given that A occurs} \]

Probability of A and B is given by

\[ P(A) \times P(B|A) \]

or, equivalently,

\[ P(B) \times P(A|B) \]

Hence, we can equate these expressions:

\[ P(A) P(B|A) = P(B) P(A|B) \]

From which follows Bayes Theorem:

\[ P(A|B) = \frac{P(B|A) P(A)}{P(B)} \]
Similarly, we Bayes Theorem for probability densities:

We have a general model

\[ y = f(q) \]

And we wish to estimate \( q \) from observations \( y \).

Let

\[ \pi(y) = \text{probability density for } y \]
\[ \pi_0(q) = \text{probability density for } q \ (\text{"prior"}) \]
\[ \pi(y|q) = \text{probability density of } y \text{ given } q \]
\[ \pi(q|y) = \text{probability density of } q \text{, given } y \]

(The "posterior")

Bayes Theorem gives

\[ \pi(q|y) = \frac{\pi(y|q) \pi_0(q)}{\pi(y)} \]

Typically we cannot easily compute \( \pi(y) \), but we can instead normalize (so \( \pi(q|y) \) is a probability density):

\[ \int \pi(q|y) \, dq = 1 \]

\[ \pi(q|y) = \frac{\pi(y|q) \pi_0(q)}{\int_{q \in Q} \pi(y|q) \pi_0(q) \, dq} \]

Where \( Q \) is the set of values \( q \) can take.

In practice this could be a high-dimensional integral — numerically integrate via "sparse grids" or Monte Carlo techniques.
Now, it amounts to characterize $\pi(y|q)$ and $\Pi_0(q)$. An example (from Smith, Section 4.8) helps explain the details.

Suppose we observe some set of coin tosses, e.g., $\text{HHTTHHTTT} \}$ $N$ tosses.

From these observations, I seek to estimate the probability $q$ of getting heads on a single coin toss.

Now, given a value of $q$ and

$N_0 = \# \text{ of observed tails in our } N \text{ tosses}$

$N_1 = \# \text{ of observed heads in our } N \text{ tosses}$

we can compute the probability of having made these observations!

$$\Pi(y|q) = q^{N_1} (1-q)^{N_0} \} \text{ Note: This is a function of } q.$$ 

Now for $\Pi_0(q)$ we might use the "uniformed prior"

$$\Pi_0(q) = 1$$

meaning that we have no special previous insight about $q$. 
If we had some prior knowledge (e.g., based on past studies of coins made by the same manufacturer) we can use something more sophisticated, like the normal distribution

\[ P_0(q) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(q-\mu)^2}{2\sigma^2}} \]

of mean \( \mu \in [0, 1] \) and variance \( \sigma^2 \). (Note that we truncate this to possible values \( q \in [0, 1] \).)

Then Bayes Theorem gives

\[ P(q|y) = \frac{P(y|q) P_0(q)}{\int_0^1 P(y|q) P_0(q) \, dq} \]

which is a function of \( q \).

\[ P(q|y) \]

the desired distribution.

See CoinToss1.m (uniformed prior)
CoinToss2.m (pool choice of prior)

on the class website.