Lecture 34: Finite Elements for the Wave Equation

34.1

1. Derive a weak form for the equation.

The weak form of the PDE is

\[ a(u,v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Gamma} f(x)v \, ds = \int_{\Omega} (u_{tt}(x,t)v(x) + (f(x)v(x) \, dx) \]

where

- \( a(u,v) \) is the energy inner product,
- \( u_{tt}(x,t) \) is the time derivative of the solution,
- \( f(x) \) is the source term,
- \( \Omega \) is the spatial domain,
- \( \Gamma \) is the boundary of the domain.

Multiply the PDE by \( v(x) \) and test function \( v \) over the domain \( \Omega \). Take the inner product of each side of the PDE with \( v \):

\[ a(u,v) = \int_{\Omega} (u_{tt}(x,t)v(x) + f(x)v(x) \, dx + \int_{\Gamma} f(x)v \, ds \]

Integrate over the spatial domain \( \Omega \). Let \( \Gamma \) be a simple smooth curve and take the inner product of each side of the PDE with \( v \):

\[ \int_{\Omega} u_{tt}(x,t)v(x) \, dx + \int_{\Omega} f(x)v(x) \, dx + \int_{\Gamma} f(x)v \, ds = \int_{\Omega} u_{xx}(x,t)v(x) \, dx + \int_{\Gamma} f(x)v \, ds \]

Initial and boundary conditions:

\[ u(t,0) = u(t,1) = 0 \]

We are solving the wave equation with finite elements, so our experience solving the heat equation with finite elements follows the usual steps.
Galerkin Approximation

Since we will struggle to find \( u \) by testing the PDE against all \( v \in C^1([0,1]) \), we instead seek

\[ U_N(x,t) \in V_N \times C[0,1] \]

for an \( N \)-dimensional subspace \( V_N = \text{span} \{ \phi_1, \ldots, \phi_N \} \) that satisfies the weak form for test functions in \( V_N \):

\[ (\frac{d^2}{dt^2} U_N, v) = -a(U_N, v) + (f, v) \quad \text{for all } v \in V_N. \]

Linear Algebra Problem

The Galerkin condition (\( \star \)) is satisfied if it holds for all vectors \( \phi_1, \ldots, \phi_N \) in a basis for \( V_N \).

We write \( U_N \in V_N \times C[0,1] \) in this basis:

\[ U_N(x,t) = \sum_{j=1}^{N} a_j(t) \phi_j(x) \]

and then require

\[ (\frac{d^2}{dt^2} U_N, \phi_k) = -a(U_N, \phi_k) + (f, \phi_k) \quad k=1, \ldots, N. \]

Expanding:

\[ (\frac{d^2}{dt^2} \sum_{j=1}^{N} a_j(t) \phi_j(x), \phi_k(x)) = -a \left( \sum_{j=1}^{N} a_j(t) \phi_j(x), \phi_k(x) \right) + (f, \phi_k) \]

\[ \Rightarrow \sum_{j=1}^{N} a_j(t) \phi_j(x), \phi_k(x) = -\sum_{j=1}^{N} a_j(t) a(\phi_j, \phi_k) + (f, \phi_k) \]

for \( k=1, \ldots, N. \)
\[ k = 1: \sum_{j=1}^{N} a_j^{(k)}(\phi_j, \phi_1) = -\sum_{j=1}^{N} a_j^{(k)}(\phi_j, \phi_1) + (f_1 \phi_1) \]

\[ k = n \quad \sum_{j=1}^{N} a_j^{(k)}(\phi_j, \phi_n) = -\sum_{j=1}^{N} a_j^{(k)}(\phi_j, \phi_n) + (f_n \phi_n) \]

Arranging in matrix form:

\[
\begin{bmatrix}
\phi_1 & \ldots & \phi_n & \\
\vdots & \ddots & \vdots & \\
\phi_1 & \ldots & \phi_n & \\
\end{bmatrix}
\begin{bmatrix}
a_1^{(n)}(t) \\
\vdots \\
a_n^{(n)}(t) \\
\end{bmatrix}
= 
\begin{bmatrix}
a_1(\phi_1, \phi_1) & \ldots & a_n(\phi_1, \phi_n) \\
\vdots & \ddots & \vdots \\
a_1(\phi_n, \phi_n) & \ldots & a_n(\phi_n, \phi_n) \\
\end{bmatrix}
\begin{bmatrix}
a_1(t) \\
\vdots \\
a_n(t) \\
\end{bmatrix}
+ 
\begin{bmatrix}
f_1 \\
\vdots \\
f_n \\
\end{bmatrix}
\]

Mass Matrix

Stiffness Matrix

Load Vector

We write this as:

\[ M a^{(n)}(t) = -K a(t) + f(t) \]

i.e.,

\[ a^{(n)}(t) = -M^{-1}K a(t) + M^{-1}f(t) \]

As in the previous lecture, we shall write this second order equation as a first order system:

\[
\begin{bmatrix}
a'(t) \\
a''(t) \\
\end{bmatrix}
= 
\begin{bmatrix}
0 & I \\
-M^{-1}K & 0 \\
\end{bmatrix}
\begin{bmatrix}
a'(t) \\
a''(t) \\
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
M^{-1}f(t) \\
\end{bmatrix}
\]

This is a \( 2N \times 2N \) matrix:

A \( 2 \times 2 \) "block matrix" with \( N \times N \) blocks.

Note how nicely this parallels the operator version in the last lecture!
Solve the linear algebraic differential equation.

First, consider the exact solution in time:

\[
\begin{bmatrix}
2(t) \\
2'(t)
\end{bmatrix} = e^{t \begin{bmatrix}
0 & I \\
-M^T & 0
\end{bmatrix}} \begin{bmatrix}
2(0) \\
2'(0)
\end{bmatrix}
+ \int_0^t e^{(t-s) \begin{bmatrix}
0 & I \\
-M^T & 0
\end{bmatrix}} f(s) \, ds
\]

What should we use for the initial conditions?

Initial position \( U_0(x) = 2(x,0) \)
Initial velocity \( V_0(x) = 2'(x,0) \)

If our finite element basis is made up of

\[
\phi_j(x_k) = \begin{cases}
1, & j = k; \\
0, & j \neq k;
\end{cases}
\]

then we can set \( 2(0) \) and \( 2'(0) \) so the solution

\[\begin{align*}
U_N(x,0) &= U_0(x) \quad \text{when } x \text{ is a grid point:} \\
U_0(x_k) &= U_N(x_k,0) = \sum_{j=1}^{N} 2_j(0) \phi_j(x_k) \\
&= 2_k(0)
\end{align*}\]

and similarly for velocity:

\[\begin{align*}
\frac{\partial}{\partial t} U_N(x,0) &= V_0(x) = \sum_{j=1}^{N} 2'_j(0) \phi_j(x_k) \\
&= 2'_k(0)
\end{align*}\]

So we set

\[
2(0) = \begin{bmatrix}
2_1(0) \\
\vdots \\
2_N(0)
\end{bmatrix} \quad 2'(0) = \begin{bmatrix}
2'_1(0) \\
\vdots \\
2'_N(0)
\end{bmatrix}
\]
Next, consider the approximate solution of the linear algebra's differential equations in time:

To simplify the notation, let
\[
\begin{bmatrix}
y(t) \\
y'(t)
\end{bmatrix} = \begin{bmatrix} z(t) \\ z'(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -M^{-1}k & 0 \end{bmatrix}, \quad g(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}
\]

Then
\[
y_k \approx y_k (k \Delta t), \quad g_k = g_k (k \Delta t)
\]

For a time step \( \Delta t \). We can advance \( y_k \) via:

- Forward Euler:
  \[
y_{k+1} = y_k + \Delta t (A y_k + g_k)
  \]
  Formulas: Euler

- or:
  \[
y_{k+1} = y_k + \Delta t (A y_{k+1} + g_{k+1})
  \]
  Backward Euler

The Backward Euler method is an implicit method (just as for the heat equation), so we find \( y_{k+1} \) via:

\[
(I - \Delta t A) \ y_{k+1} = y_k + \Delta t \ g_{k+1}
\]

\[
\Rightarrow \quad y_{k+1} = (I - \Delta t A)^{-1} (y_k + \Delta t \ g_{k+1})
\]

Backward Euler.

Of course, these are just two of many options. How will they behave as \( k \to \infty \)? That will depend on properties of the method, and the eigenvalues of
\[
A = \begin{bmatrix} 0 & I \\ -M^{-1}k & 0 \end{bmatrix}
\]