Lecture 30: Stability of Time Integration

We seek to understand the apparent time step restriction for the Forward Euler method for the heat equation.

First consider a generic scalar differential equation

\[ y'(t) = \lambda y(t) \]

for which the Forward Euler method takes the form

\[ y_{kn} = y_k + \Delta t \lambda y_k = (1 + \Delta t \lambda) y_k \]

So that

\[ y_k = (1 + \Delta t \lambda)^k y_0. \]

Now the exact solution of the differential equation is

\[ y(t) = e^{\lambda t} y(0), \]

which converges, \( y(t) \to 0 \), provided \( \text{Re}(\lambda) < 0 \), i.e., if \( \lambda = \alpha + i \beta \), then

\[ e^{\lambda t} = e^{(\alpha + i \beta) t} = e^{\alpha t} e^{i \beta t} = e^{\alpha t} (\cos \beta t + i \sin \beta t) \]

So

\[ |e^{\lambda t}| = |e^{\alpha t} (\cos \beta t + i \sin \beta t)| = e^{\alpha t} |\cos \beta t + i \sin \beta t| = e^{\alpha t} \sqrt{\cos^2 \beta t + \sin^2 \beta t} = e^{\alpha t}. \]

So \( e^{\lambda t} \to 0 \) if \( \text{Re}(\lambda) = \alpha < 0 \).
Does Forward Euler mimic this behavior? 30.

\[ y_k = (1 + \Delta t \lambda)^k y_0 \]

Implies that \( y_k \to 0 \) provided \( |1 + \Delta t \lambda| < 1 \).

Note that this is a rather different condition than \( \text{Re}(\lambda) < 0 \)!

Consider real values of \( \lambda \), as we have for eigenvalues of the heat discretization. If \( \lambda \) is negative but large in magnitude, we will need \( \Delta t \) to be very small for \( |1 + \Delta t \lambda| < 1 \).

Here is a helpful interpretation of this condition:

\[ \{ z \in \mathbb{C} : |1 - z| < 1 \} = \text{set of all complex numbers } z \]

for which \( |1 - z| < 1 \)

= \text{set of all complex numbers } z \]

whose distance from 1 is less than one.

Thus \( |1 + \Delta t \lambda| = |1 - \Delta t \lambda| < 1 \) is satisfied if \( \Delta t \lambda \) is in the disc of radius 1 in the complex plane, centered at -1.

\[ \text{we need } \Delta t \lambda \text{ to be in this disc, for } y_k \to 0. \]

\[ \text{Real} \]

\[ \text{Imag} \]
Now let us interpret this observation in light of the eigenvalue calculation in the last lecture.

The best equation/Finite Element method give

$$a'(t) = -M^{-1}K a(t)$$

So Forward Euler takes the form

$$a_{j+1} = a_j - \Delta t M^{-1} K a_j \Rightarrow a_j = (I - \Delta t M^{-1} K)^j a_0$$

where $$a_j \approx a(j \Delta t)$$.

In the last lecture we wrote

$$M^{-1}K = V V^T = \sum_{k=1}^{n} \lambda_k V_k V_k^T$$

where

$$\lambda_k = \frac{6}{h^2} \left( \frac{2 - \gamma_k}{4 + \gamma_k} \right), \quad \gamma_k = 2 \cos(k \pi/N+1), \quad h = \frac{1}{N+1}.$$ 

Forward Euler then becomes

$$a_j = (I - \Delta t M^{-1} K)^j a_0$$

$$= (I - \Delta t V V^T)^j a_0$$

$$= (V V^T - \Delta t V V^T)^j a_0$$

$$= V (I - \Delta t \Lambda)^j V^T a_0$$

$$= V \begin{pmatrix} \lambda_1 \quad \ddots & 0 & \ddots & \vdots \\ 0 & \ddots & \lambda_2 & \vdots \\ \vdots & \ddots & \ddots & \lambda_j \end{pmatrix}^j V^T a_0$$

$$= \sum_{k=1}^{N} (1 - \Delta t \lambda_k)^j V_k V_k^T a_0$$

Note:

$$V V^T = I$$

$$V^T V = I$$

The component of this sum that grows fastest is the one for which $|1 - \Delta t \lambda_k|$ is largest.

This corresponds to $\lambda_N$ when $\Delta t$ violates the stability condition $\Rightarrow$ the unstable solution will be dominated by $V_N$, the "smallest" eigenvector.
So if we want $a_j \to 0$ as $j \to \infty$, we need $|1 - \Delta t \lambda_k| < 1$ for all $k = 1, \ldots, N$.

Since $\lambda_k \in (0, \frac{12}{h^2})$ for all $k = 1, \ldots, N$, we should pick $\Delta t$ sufficiently small that

$$ |1 - \Delta t \frac{12}{h^2}| < 1 $$

Since $\Delta t \frac{12}{h^2} > 0$, for any $\Delta t$, $|1 - \Delta t \frac{12}{h^2}| < 1$.

Hence, $\Delta t$ will be determined by the condition

$$ -1 < 1 - \Delta t \frac{12}{h^2} $$

$$ \Rightarrow \Delta t \frac{12}{h^2} < 2 \Rightarrow \boxed{\Delta t < \frac{h^2}{6}} $$

Notice the profound implication of this calculation:

If you want to double $N$ (to get better accuracy in space), you reduce $h$ by $\frac{1}{2}$, and you must then reduce $\Delta t$ by $\frac{1}{4}$.

This severe stability constraint is called the **CFL condition** after its discoverers: Richard Courant, Kurt Friedrichs, and Hans Lewy.
Each different time integrator leads to a different stability constraint.

As an example, consider the backward Euler method. For the generic equation \( y'(t) = \lambda y(t) \), note that

\[
y'(t) = \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t}
\]

Also suggests that

\[
0 = \lim_{\Delta t \to 0} \frac{y'(t+\Delta t) - (y(t+\Delta t) - y(t))}{\Delta t}
\]

Hence, for finite \( \Delta t \), approximate

\[
y(t+\Delta t) \approx y(t) + \Delta t y'(t+\Delta t)
\]

Writing \( y_j \approx y(j\Delta t) \), this gives

\[
y_{j+1} = y_j + \Delta t \lambda y_{j+1} \quad (y' = \lambda y)
\]

\[
\Rightarrow (1-\Delta t\lambda) y_{j+1} = y_j
\]

\[
\Rightarrow y_{j+1} = \frac{1}{1-\Delta t\lambda} y_j
\]

And so we can accumulate \( j \) steps as

\[
y_j = \left( \frac{1}{1-\Delta t\lambda} \right)^j y_0
\]

Now if \( \text{Re}(\lambda) < 0 \), \( y(t) \to 0 \). Is the same true for backward Euler?
\[ y_j \to 0 \text{ if } \left| \frac{1}{1 - \Delta t \lambda} \right| < 1, \]  which implies \[ \lambda \Delta t < 1. \]

\[ |1 - \Delta t \lambda| = \text{Set of all } \Delta t \lambda \text{ that are a distance greater than 1 from 1 in the complex plane} \]

So \( y_j \to 0 \) provided \( \Delta t \lambda \) is in the shaded region.

If \( \text{Re}(\lambda) < 0 \), then for any \( \Delta t \),
\[ \text{Re}(\Delta t \lambda) < 0 \], and so \( \Delta t \lambda \) is in the shaded region.

\[ \Rightarrow \text{Backward Euler is stable for any } \Delta t. \]

(You will still adjust \( \Delta t \) to give an accurate solution, i.e., to make the approximation \( y' \approx \frac{y(t + \Delta t) - y(t)}{\Delta t} \) accurate.)

For the heat equation, \( \Delta'(t) = -M^{-1}K \Delta(t) \),
Backward Euler gives
\[ \Delta_j = (I + \Delta t M^{-1}K)^j \Delta_0 = \sum_{k=1}^{N} \left( \frac{1}{1 + \Delta t \lambda_k} \right)^j V_k V_k^T \Delta_0 \]
\( \lambda_k \in (0, \frac{12}{h^2}) \)

Since \( \lambda_k > 0 \), \[ \left| \frac{1}{1 + \Delta t \lambda_k} \right| < 1 \text{ for all } \Delta t. \]
Hence Backward Euler is stable for the heat equation for any $\Delta t \geq 0$.

Coda: A word about time-stepping implementations
(See Lecture 35 for related detail.)

**Forward Euler:**

$$a_{j+1} = a_j - \Delta t M^{-1}K a_j$$

**Backward Euler:**

$$a_{j+1} = (I + \Delta t M^{-1}K)^{-1} a_j$$

$$= (M^{-1}M + \Delta t M^{-1}K)^{-1} a_j$$

$$= (M^{-1} (M + \Delta t K))^{-1} M^{-1} a_j$$

$$= (M + \Delta t K)^{-1} M a_j$$

Both equations involve the inverse of one matrix, which can be implemented via LU factorization. Similar work! per step.

For more complicated equations (especially nonlinear equations) Backward Euler is rather more expensive per step than Forward Euler.