All of the problems on this set use the inner product
\[(u, v) = \int_0^1 u(x)v(x) \, dx.\]

1. [20 points]
On Problem Set 3, Question 1(d), we sought the solution of the equation \(-u''(x) = f(x),\) \(u(0) = u(1) = 0,\) where
\[f(x) = \begin{cases} 1, & 0 \leq x < 1/2; \\ 0, & 1/2 < x \leq 1. \end{cases}\]
The exact solution to this problem is:
\[u(x) = \begin{cases} 3x/8 - x^2/2, & 0 \leq x < 1/2; \\ (1 - x)/8, & 1/2 < x \leq 1. \end{cases}\]
You can implement \(u\) in Chebfun via the commands:
\[
x = \text{chebfun}('x',[0 1]); \\
u = \text{chebfun}([3*x/8-(x.^2)/2,(1-x)/8],[0 .5 1],'splitting','on');
\]
Throughout this problem, let \(\phi_1, \ldots, \phi_N\) denote the usual hat functions for a grid with uniform spacing equal to \(h = 1/(N + 1).\)

(a) Suppose that \(N\) is odd. Compute by hand the inner products \((f, \phi_j)\) for \(j = 1, \ldots, N.\)
(b) Modify the \texttt{fem1.m} code on the class website (or write your own) to construct approximate solutions \(u_N\) to this equation via the finite element method on the subspace \(V_N = \text{span}\{\phi_1, \ldots, \phi_N\}.\)
Produce plots comparing the exact solution \(u(x)\) to the approximate solution \(u_N(x)\) for \(0 \leq x \leq 1\) for \(N = 3, 7, 15, 31, 63.\)
(c) Produce plots of the error \(u(x) - u_N(x)\) for \(N = 3, 7, 15, 31, 63.\) (These may all be on one plot.)

2. [30 points]
On Problem Set 3, Question 3, you sought solutions to the equation \(-u''(x) + 7u(x) = 1\) with \(u(0) = u(1) = 0,\) The problem considers the finite element method for this equation.

(a) Suppose that \(u\) satisfies the differential equation. Show that for all \(v \in C^2_D[0, 1],\) the weak form holds:
\[a(u, v) = (f, v),\]
where \(a(u, v) = \int_0^1 (u'(x)v'(x) + 7u(x)v(x)) \, dx\) and \((f, v) = \int_0^1 f(x)v(x) \, dx.\)
(b) We now wish to approximate the solution to this equation using the finite element method with standard hat functions \(\phi_1, \ldots, \phi_N\) on a grid with uniform spacing, \(h = 1/(N + 1).\)
Calculate (by hand) the entries in the stiffness matrix \(K,\) i.e., \(K_{j,k} = a(\phi_j, \phi_k),\) for the energy inner product in part (a).
Hint: you can write your answer in terms of the stiffness matrix we obtained for \(-u''(x) = f(x),\) which we shall call \(K_0\) here, and the Gram matrix for hat functions, which we shall call \(M\) here:
\[
K_0 = \frac{1}{h} \begin{bmatrix} 2 & -1 & \cdots & -1 \\ -1 & 2 & \cdots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & 2 & \cdots \\ \cdots & \cdots & \cdots & \ddots \\ \end{bmatrix}, \\
M = h \begin{bmatrix} 2/3 & 1/6 \\ 1/6 & 2/3 \\ \vdots & \ddots \\ \vdots & \ddots & \ddots \\ 1/6 & 2/3 \\ \end{bmatrix},
\]
where unspecified entries are zero.
(c) Use your new stiffness matrix to construct approximate solutions $u_N$ to $-u''(x) + 7u(x) = f(x)$ with $u(0) = u(1) = 0$ for $N = 2, 4, 8, 16$. Produce a plot showing your approximation solutions $u_N$ and the exact solution, which can be implemented in Chebfun via the commands:

```matlab
x = chebfun('x', [0 1]);
uf = (1+exp(sqrt(7))-exp(-sqrt(7)*(x-1))-exp(sqrt(7)*x))/(7*(1+exp(sqrt(7))));
```

(d) Produce plots showing the error $u(x) - u_N(x)$ for $N = 2, 4, 8, 16$. (These may all be on one plot.)

3. [30 points]

In class we have derived the weak form of the differential equation $-u''(x) = f(x)$ with (a) homogeneous Dirichlet boundary conditions, $u(0) = u(1) = 0$, and (b) homogeneous Neumann boundary conditions, $u'(0) = u'(1) = 0$. Recall that the Dirichlet boundary conditions were essential, meaning that we had to use test functions that satisfy this boundary conditions, $v \in C^2_\mathbb{D}[0,1]$, while the Neumann boundary conditions were natural, meaning that the test functions did not need to include the boundary conditions, $v \in C^2[0,1]$. (They key issue was making sure the boundary term that arose during integration by parts is zero.)

(a) Derive the weak form of the differential equation $-u''(x) = f(x)$ for the mixed boundary conditions $u'(0) = u(1) = 0$. That is, show that if $-u''(x) = f(x)$, then $a(u, v) = (f, v)$. Specify the energy inner product $a(u, v)$, and be very clear about what space the test functions $v$ should come from.

(b) We wish to construct finite element approximate solutions to $-u''(x) = x$ with $u'(0) = u(1) = 0$, using hat functions on a grid with uniform spacing $h = 1/(N+1)$. In particular, we will use the approximation space

$$V_N = \text{span}\{\phi_0, \phi_1, \ldots, \phi_N\}.$$ 

Write out the entries of the stiffness matrix $K \in \mathbb{R}^{(N+1)\times(N+1)}$.

(c) Write MATLAB code to compute the finite element approximations $u_N$ from $V_N$, and produce a plot comparing the solutions $u_N(x)$ to the exact solution $u(x) = (1 - x^3)/6$, for $N = 2, 4, 8, 16$. (You may plot the errors $u(x) - u_N(x)$, if the approximate solutions are difficult to tell apart.)

(d) Do the approximate solutions $u_N$ satisfy the boundary conditions exactly? That is, does $u_N(0) = 0$? Does $u_N(1) = 0$?

4. [20 points]

We wish to solve the differential equation

$$-(\kappa(x)u'(x))' = f(x)$$

with $0 \leq x \leq 1$ and $\kappa(x) = e^x$, subject to homogeneous Dirichlet boundary conditions:

$$u(0) = u(1) = 0.$$ 

With this simple coefficient $\kappa(x) = e^x$ the eigenfunctions would take an intricate form, thus making the spectral method difficult to implement. Thus this problem is a prime candidate for solution with the finite element method. The weak form of the problem is:

$$a(u, v) = (f, v)$$

for all $v \in C^2_\mathbb{D}[0,1]$, where $a(u, v) = \int_0^1 \kappa(x)u'(x)v'(x) \, dx$.

For this problem, we shall use $f(x) = x$, giving the exact solution

$$u(x) = \frac{(1-e)x^2 + (2-2e)x + 3(e^x - 1)}{2(1-e)e^x}.$$ 

As usual, we shall look for finite element solutions from the span of hat functions,

$$V_N = \text{span}\{\phi_1, \ldots, \phi_N\}.$$
(a) Write down a formula for $a(\phi_j, \phi_\ell)$, the $(j, \ell)$ entry of the stiffness matrix $K$. (Hint: your answer can depend on the grid points $x_j = jh$.)

(b) Write MATLAB code to compute the finite element approximation $u_N$ from $V_N$, and produce a plot showing the solutions $u_N(x)$ for $N = 4, 8, 16, 32$.

(c) Write MATLAB code to plot the error $u(x) - u_N(x)$ for $N = 4, 8, 16, 32$. 