Saturday 10 May 2014.

Instructions:

1. Time limit: **2 uninterrupted hours**.

2. There are three questions worth a total of 100 points.

3. You *may not* use any outside resources, calculators, or MATLAB.

4. Please answer the questions thoroughly and justify all your answers. 
   *Show all your work to maximize partial credit.*

5. Print your name on the line below:

   ______________________________________

6. Time started: ________________  Time completed: ________________
1. [30 points: 6 points per part]

(a) Consider a discrete-time dynamical system \( x_{k+1} = B x_k \) with solution \( x_k = B^k x_0 \), where
\[
B = \sum_{j=1}^{n} \lambda_j v_j \hat{v}_j^*,
\]
is an eigenvalue decomposition of \( B \).
State the conditions under which \( \| x_k \| \to 0 \) as \( k \to \infty \).

(b) Suppose we have some matrix \( A \) with eigenvalue–eigenvector pair \( (\lambda, v) \), so \( A v = \lambda v \).
Show that \( v \) is also an eigenvector of \( C = I + \frac{1}{2} A \). What is the corresponding eigenvalue of \( C \)?

The Jacobi method is a very simple iterative method for solving \( A x = b \). Here is how it works. Let \( D \) be a matrix that is zero everywhere, except that its main diagonal is the same as the main diagonal of \( A \):
\[
D = \text{diag}(a_{1,1}, a_{2,2}, \ldots, a_{n,n}).
\]
The Jacobi method constructs vectors \( x_k \) that approximate the solution \( x \) via the recurrence
\[
x_{k+1} = D^{-1}((D - A)x_k + b),
\]
starting from some initial guess \( x_0 \).

(c) Show that if \( x_0 \) is the exact solution, \( x_0 = x \), where \( A x = b \), then \( x_k = x \) for all \( k \).

(d) Let \( r_k = b - A x_k \) be the \( k \)th residual of the Jacobi method. Use the definition of \( x_{k+1} \) to show that \( r_{k+1} = b - A x_{k+1} \) satisfies
\[
r_{k+1} = (\mathbf{I} - AD^{-1})r_k.
\]
(Notice that this implies \( r_{k+1} = (\mathbf{I} - AD^{-1})^k r_0 \).

(e) Now we revisit a special (but very important) matrix that appeared on Problem Set 4,
\[
A = \begin{bmatrix}
-2 & 1 \\
1 & -2 & 1 \\
& 1 & -2 & \ddots & \ddots \\
& & \ddots & \ddots & 1 \\
& & & 1 & -2
\end{bmatrix} \in \mathbb{C}^{n \times n}
\]
whose diagonal is simply \( D = -2 \mathbf{I} \) and whose eigenvalues are
\[
\lambda_j = -2 + 2 \cos \left( \frac{j \pi}{n+1} \right), \quad j = 1, \ldots, n.
\]
Using the preliminary parts of this problem, show that when you apply the Jacobi method to this \( A \), you always have \( r_k \to 0 \) for any initial guess \( r_0 \) and any dimension \( n \).
2. [36 points: 6 points per part]

At some point in your past, you learned that the dot product describes the angle between two vectors: For nonzero \( x, y \in \mathbb{R}^n \),
\[
x^T y = \cos \angle(x, y) \|x\| \|y\|,
\]
which can be regrouped as
\[
\cos \angle(x, y) = \left( \frac{x}{\|x\|} \right)^T \left( \frac{y}{\|y\|} \right).
\]

Here we explore how to generalize this idea to understand the angles between two subspaces.

(a) Consider the two 2-dimensional subspaces
\[
X = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad Y = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.
\]

Each of these subspaces can be visualized as a plane in \( \mathbb{R}^3 \). What is their intersection? Describe, in geometrical language, how are these planes arranged relative to one another.

We define the angles between two \( d \)-dimensional subspaces \( X \) and \( Y \) of \( \mathbb{C}^n \) as follows:

- Construct a matrix \( Q_X \in \mathbb{C}^{n \times d} \) whose columns form an orthonormal basis for \( X \);
- Construct a matrix \( Q_Y \in \mathbb{C}^{n \times d} \) whose columns form an orthonormal basis for \( Y \);
- Form the matrix \( A = Q_X^* Q_Y \);
- Compute the singular values \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_d \) of \( A \).
- We define the \( j \)th smallest angle \( \theta_j(X, Y) \) between \( X \) and \( Y \) to be
\[
\cos \theta_j(X, Y) = \sigma_j.
\]

(b) Explain how this formula reduces to (*) for the case \( d = 1 \).

(c) Compute \( \theta_1(X, Y) \) and \( \theta_2(X, Y) \) for the subspaces \( X \) and \( Y \) in part (a). Interpret this result in light of your answer to part (a). Does this generalized notion of the angle between subspaces agree with your geometric interpretation?

(d) Compute \( \theta_1(X, Z) \) and \( \theta_2(X, Z) \), where \( X \) is again from part (a) and
\[
Z = \text{span} \left\{ \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.
\]

(e) Now suppose \( X \) and \( Y \) are general \( d \)-dimensional subspaces, and let \( u_j \) and \( v_j \) denote the \( j \)th left and right singular vectors of \( A \). Using the definition (*), what is the angle between the vectors \( Q_X u_j \) (which is in \( X \)) and \( Q_Y v_j \)?

(f) Suppose the first \( p \) singular values of \( Q_X^* Q_Y \) are equal to one, \( \sigma_1 = \cdots = \sigma_p = 1 \). Describe the vectors at the intersection of \( X \) and \( Y \).
3. [34 points: (a)=10 points; (b)=6; (c)=6; (d)=6; (e)=6 points]

Suppose \( A \in \mathbb{C}^{m \times n} \) is a rank-\( r \) matrix with singular value decomposition
\[
A = U\Sigma V^* = \sum_{j=1}^{r} \sigma_j u_j v_j^*,
\]
where \( U = [u_1 \cdots u_m] \in \mathbb{C}^{m \times m} \) and \( V = [v_1 \cdots v_n] \in \mathbb{C}^{n \times n} \), and \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 \).

Recall that pseudoinverse is given by
\[
A^+ = \sum_{j=1}^{r} \frac{1}{\sigma_j} v_j u_j^*.
\]

(a) Consider the matrix
\[
A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.
\]

What is the rank of \( A \)? Compute the pseudoinverse \( A^+ \).

(b) Does the equation \( Ax = b \),
\[
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix},
\]
have any solutions? If so, compute all of them. If not, find any/all solutions of
\[
\min_{x \in \mathbb{C}^2} \|b - Ax\|.
\]

In either case, if you have multiple solutions, be sure to specify which of those solutions minimizes \( \|x\| \).

(c) Suppose we make a small change to the \((1,1)\) entry of \( A \). Describe, in as much detail as possible (but no computations are required) what such a change will do to the rank, singular values, pseudoinverse, and solution to \( Ax = b \) or \( \min_{x \in \mathbb{C}^2} \|b - Ax\| \) of smallest norm.

(d) Consider now the regularized problem
\[
\min_{x \in \mathbb{C}^2} \left\| \begin{bmatrix} A \\ \lambda I \end{bmatrix} x - \begin{bmatrix} b \\ 0 \end{bmatrix} \right\|.
\]

Solve this problem for the \( A \) and \( b \) in part (b) with regularization parameter \( \lambda = 1 \).

(e) Recall that, in general, the regularized solution takes the form
\[
x_\lambda = \sum_{j=1}^{n} \frac{\sigma_j}{\sigma_j^2 + \lambda^2} (u_j^* b)v_j.
\]

For the \( A \) and \( b \) in part (b), describe how \( x_\lambda \) will behave as \( \lambda \to 0 \) and \( \lambda \to \infty \).

If we make the alteration to the \((1,1)\) entry described in part (b), will this affect the behavior of \( x_\lambda \) as \( \lambda \to 0 \) and/or \( \lambda \to \infty \)?