Thursday 20 March 2014.

Instructions:

1. Time limit: 2 uninterrupted hours.

2. There are three questions worth a total of 100 points.

3. You may not use any outside resources, calculators, or MATLAB.

4. Please answer the questions thoroughly and justify all your answers.
   *Show all your work to maximize partial credit.*

5. Print your name on the line below:

   ________________________________________________________________

6. Time started: ____________________  Time completed: ____________________
1. [28 points: 7 points per part]

Consider this circuit, a two-compartment model of a neuron where the current flow is resisted by intercellular material ($R_i$), the cell membrane ($R_m$), and a myelin sheath that imposes both transverse resistance ($R_s$) and axial resistance ($R_a$).

Please work through the four modeling steps outlined in class for this circuit. Be sure to specify all entries in the matrices and vectors.

(a) Write the voltage drops, $e$, in terms of the potentials, $x$, as $e = v - A x$.

(b) Apply Ohm’s Law to write the currents, $y$, in terms of the voltage drops, $e$, as $y = Ke$.

(c) Express Kirchhoff’s Current Law via $A^t y = 0$.

(d) Compute $A^t KA$.

To expedite this calculation, you may use that (1) $A^t KA$ is a symmetric matrix; and (2) premultiplication by the diagonal matrix $K$ scales the rows:

$$KA = \begin{bmatrix} k_1 & & k_2 & & k_m \\ & \ddots & & \ddots & \\ & & k_m \end{bmatrix} \begin{bmatrix} \text{ROW 1} \\ \text{ROW 2} \\ \vdots \\ \text{ROW m} \end{bmatrix} = \begin{bmatrix} k_1 \times \text{ROW 1} \\ k_2 \times \text{ROW 2} \\ \vdots \\ k_m \times \text{ROW m} \end{bmatrix}.$$

[from Steve Cox]
2. [40 points: (a)=6; (b)=4; (c)=4; (d)=6; (e)=12; (f)=4; (g)=4 points]

(a) State the Fundamental Theorem of Linear Algebra for the matrix $A \in \mathbb{C}^{m \times n}$. (Your answer should refer to each of the spaces $\mathcal{R}(A)$, $\mathcal{R}(A^*)$, $\mathcal{N}(A)$, and $\mathcal{N}(A^*)$, and should also comment on orthogonality of these spaces.)

(b) Suppose we wish to solve the equation $Ax = f$ for unknown $x$. Under what condition on $f$ will there exist a solution? When a solution exists, under what condition will it be unique?

(c) Suppose $A \in \mathbb{C}^{4 \times 5}$. Considering the size of $A$ alone: what are the maximum and minimum dimensions of $\mathcal{R}(A)$, $\mathcal{R}(A^*)$, $\mathcal{N}(A)$, and $\mathcal{N}(A^*)$?

For the rest of the problem, consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 0 & 2 & 0 \end{bmatrix}$$

(d) Compute an echelon form $A_{\text{red}}$ of $A$. What are the pivot rows? What are the pivot columns?

(e) Compute bases for the four fundamental subspaces, $\mathcal{R}(A)$, $\mathcal{R}(A^*)$, $\mathcal{N}(A)$, and $\mathcal{N}(A^*)$.

(f) Suppose we want to write down all solutions to $Ax = f$, with $f = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Here is one solution $x_R$ that satisfies $Ax_R = f$:

$$x_R = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(Notice that $x_R \in \mathcal{R}(A^*)$.)

Write down an expression for infinitely many solutions $x$ to $Ax = f$ for this $f$. Write these solutions as $x = x_R + v$, where you should specify a subspace from which $v$ comes.

(g) Of the infinitely many solutions identified in part (f), which one minimizes $\|x\| = \sqrt{x^*x}$? Explain.
3. [32 points: (a)=8; (b)=5; (c)=5; (d)=7; (e)=7 points]

(a) Suppose $B$ is a diagonalizable matrix with spectral representation $B = \sum_{j=1}^{n} \phi_j P_j$ and eigenvalues $\phi_1, \ldots, \phi_n$, and let $x^{(k)} = B^k x(0)$.

(i) How will $\|x^{(k)}\|$ behave as $k \to \infty$ if all eigenvalues of $B$ satisfy $|\phi_j| < 1$?

(ii) Suppose $|\phi_1| > 1$ and $|\phi_j| < 1$ for $j = 2, \ldots, n$. How will $\|x^{(k)}\|$ behave as $k \to \infty$?

Our study of eigenvalues was motivated by analysis of the dynamical system

$x'(t) = Ax(t), \quad x(0) = x_0.$

Suppose $A$ is a diagonal matrix and, for simplicity, assume all its eigenvalues are real, negative numbers. Hence $e^{tA} \to 0$ as $t \to \infty$ for all eigenvalues $\lambda$ of $A$, and so $x(t) \to 0$ as $t \to \infty$. Such a system is stable.

In class we saw the solution $x(t)$ is given via the matrix exponential: $x(t) = e^{tA}x(0)$. However, this approach is computationally expensive. In most applications we settle for a quicker approximation that only requires us to compute matrix-vector products.

In this problem we study two approaches. If $\alpha \in \mathbb{R}$ is a scalar, recall from calculus that

$$e^{t\alpha} = \lim_{h \to 0} (1 + h\alpha)^{t/h} = \lim_{h \to 0} \left( \frac{1}{1 - h\alpha} \right)^{t/h}.$$

The forward and backward Euler methods build similar approximations to $e^{tA}$. For some fixed time-step $h > 0$:

$$e^{tA} \approx (I + hA)^{t/h} \quad \text{ (forward Euler)}$$

$$e^{tA} \approx \left( (I - hA)^{-1} \right)^{t/h} \quad \text{ (backward Euler)}.$$

Thus at time $t_k = hk$, we approximate $x^{(k)} \approx x(t_k) = x(kh)$ via

$$x^{(k)} = (I + hA)^k x(0) \quad \text{ (forward Euler)}$$

$$x^{(k)} = ((I - hA)^{-1})^k x(0) \quad \text{ (backward Euler)}.$$

(b) The matrix $(I + hA)$ has the same eigenvectors as $A$. What are its eigenvalues, in terms of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A$?

(c) The matrix $(I - hA)^{-1}$ has the same eigenvectors as $A$. What are its eigenvalues, in terms of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A$?

Suppose the eigenvalues of $A$ satisfy $\lambda_1 \leq \cdots \leq \lambda_n < 0$, which ensures $x(t) = e^{tA}x(0) \to 0$ as $t \to \infty$. We want to see if the forward and backward Euler approximations capture the same behavior, i.e., if $x^{(k)} \to 0$ as $k \to \infty$.

(d) First consider the backward Euler method. In this case, $x^{(k)} \to 0$ for any choice of $h$.

Explain why this is the case, in light of the eigenvalues of $(I - hA)^{-1}$.

(e) For the forward Euler method, choosing $h$ too large will cause $\|x^{(k)}\| \to \infty$! Describe how $h$ must relate to the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A$ in order to ensure that $x^{(k)} = (I + hA)^k x(0) \to 0$ as $k \to \infty$. 

3