1. [25 points: 10 points for (a); 5 points each for (b),(c),(d)]

Consider the vectors
\[ a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 4 \\ 2 \\ 5 \\ 2 \end{bmatrix}. \]

(a) Perform (by hand) the Gram–Schmidt orthogonalization procedure on \{a_1, a_2, a_3\} to produce orthonormal vectors \( q_1, q_2, \) and \( q_3 \) such that \( \text{span} \{q_1, q_2, q_3\} = \text{span} \{a_1, a_2, a_3\} \).

(b) You can arrange the quantities you have computed in the Gram–Schmidt process into a QR factorization of the matrix \( A = [a_1 \ a_2 \ a_3] \): Define \( Q = [q_1 \ q_2 \ q_3] \) and define \( R \) entrywise as
\[
r_{j,k} = \begin{cases} 
q_j^T a_k, & j < k; \\
\|a_k - \sum_{\ell=1}^{k-1} q_j^T a_\ell q_\ell\|, & j = k; \\
0, & j > k.
\end{cases}
\]
Write out \( Q \) and \( R \), and confirm that \( QR = A \).

(c) Compute \( QQ^T \) (by hand) and \( AA^+ = A(A^T A)^{-1} A^T \) (in MATLAB) and confirm that they are equal. (Both formulas give the orthogonal projector onto \( \mathcal{R}(A) \).)

(d) Consider the least squares problem
\[
\min_{x \in \mathbb{R}^3} \|b - Ax\|, \quad b = \begin{bmatrix} 3 \\ -5 \\ -1 \\ 1 \end{bmatrix}.
\]

In MATLAB, compute the least squares solution in two ways:
\[
x = R^{-1}Q^T b, \quad x = (A^T A)^{-1} A^T b,
\]
and confirm that you get the same answer.

2. [25 points: 12 points for (a); 3 points for (b)]

Matrices with orthonormal columns play an important role in this course – and in many applications. Special rotation matrices arise often in computer graphics and dynamics.
(a) Given an angle \( \theta \), define the rotation matrix
\[
Q_\theta := \begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix}.
\]
Prove that \( Q_\theta \) has orthonormal columns for any value of \( \theta \), that is, \( Q_\theta^T Q_\theta = I \).

(b) The class website contains the file hw5.mat. Get this file and load hw5 to have access to the variable \( X \in \mathbb{R}^{2 \times 48} \), which contains data points in the \((x,y)\) plane, stored in the form
\[
X = \begin{bmatrix}
x_{1} & x_{2} & \cdots & x_{48} \\
y_{1} & y_{2} & \cdots & y_{48}
\end{bmatrix}.
\]
Visualize this data set via
\[
\text{plot}(X(1,:),X(2,:),'k.','markersize',14)
\]
Be sure to use axis equal to optimize viewing.

(c) For the 8 values \( \theta = \pi/4, \pi/2, 3\pi/4, \ldots, 2\pi \):
(i) construct \( Q_\theta \) in MATLAB
(ii) using a command similar to the one in part (b), plot \( Q_\theta X \) in the \((x,y)\) plane.
Use axis equal and hold on to superimpose all 8 plots into one.

(d) The file hw5.mat also contains another variable, \( Y \in \mathbb{R}^{3 \times 7190} \), that contains a point cloud in three dimensions (derived from the Stanford 3D Scanning Repository):
\[
Y = \begin{bmatrix}
x_{1} & x_{2} & \cdots & x_{7190} \\
y_{1} & y_{2} & \cdots & y_{7190} \\
z_{1} & z_{2} & \cdots & z_{7190}
\end{bmatrix}.
\]
Given angles \( \theta \) and \( \phi \), define the two rotation matrices
\[
Q_{xy} := \begin{bmatrix}
\cos(\theta) & -\sin(\theta) & 0 \\
\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad Q_{yz} := \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(\phi) & -\sin(\phi) \\
0 & \sin(\phi) & \cos(\phi)
\end{bmatrix}.
\]
Set \( \theta = 3\pi/4 \) and \( \phi = \pi/2 \).
Create one plot showing four versions of this data set:
\[
Y, \quad Q_{xy} Y, \quad Q_{yz} Y, \quad Q_{xy} Q_{yz} Y.
\]
To show this three-dimensional data, use the \texttt{plot3} command, e.g.,
\[
\text{plot3}(Y(1,:),Y(2,:),Y(3,:),'k.'){\textcolor{red}{,}}'markersize',14)
\]
(Plot each data set as a different color; use axis equal.)
Describe the effect of each of the matrices \( Q_{xy} \) and \( Q_{yz} \) on the data set.

3. [25 points: 15 points for (a); 5 points each for (b),(c)]

Arguably the most fundamental matrix in applied mathematics has the simple form
\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots \\
1 & 0 & 1 & \cdots \\
0 & 1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \cdots \\
1 & \cdots & 0 & 0
\end{bmatrix},
\]
a symmetric matrix with ones on the first super-diagonal and sub-diagonal, with zeros everywhere else.
(a) Let $A$ be an $N \times N$ matrix of this form. The eigenvalues $\lambda_j$ and eigenvectors $v_j$ have the elegant form

$$
\lambda_j = 2 \cos \left( \frac{j \pi}{N+1} \right), \quad v_j = \begin{bmatrix}
\sin \left( \frac{j \pi}{N+1} \right) \\
\sin \left( \frac{2j \pi}{N+1} \right) \\
\vdots \\
\sin \left( \frac{Nj \pi}{N+1} \right)
\end{bmatrix}
$$

for each $j = 1, \ldots, N$.

Verify that these formulas do indeed give an eigenvalue–eigenvector pair for $A$, for $j = 1, \ldots, N$. (To do so, compute $Av_j$ and $\lambda_j v_j$, and show that all entries of these two vectors are the same.) Hint: Recall that $2 \cos(\phi) \sin(\theta) = \sin(\theta + \phi) + \sin(\theta - \phi)$.

(b) For $N = 32$, plot the eigenvectors of $j = 1, 2, 31, 32$.

(Plot the entry number $k = 1, \ldots, 32$ along the horizontal axis, and the entries $(v_j)_k$ of the eigenvector along the vertical axis. If $v_j$ is a vector of length 32 containing the entries of $v_j$, then

```matlab
plot([1:32],v_j,'k.-','markersize',24,'linewidth',1)
```

will produce a nice plot. Produce a distinct plot for each of $j = 1, 2, 31, 32$.)

(c) Arrange the 32 vectors $v_j$ into a matrix $V = [v_1 \ v_2 \ \cdots \ v_N]$.

Check the orthogonality of the eigenvectors by computing all their dot products:

```matlab
imagesc(V'*V), colorbar
```

Explain how this plot reveals the orthogonality of the eigenvectors.

---

4. [25 points: 6 points for (a),(c),(d); 7 points for (b)]

Consider the matrix

$$
A = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
-1 & 1
\end{bmatrix}.
$$

(a) Compute the eigenvalues and eigenvectors of $A^T A$.

(b) Write down a singular value decomposition, $A = U \Sigma V^T$, where $U \in \mathbb{R}^{3 \times 2}$ and $V \in \mathbb{R}^{2 \times 2}$.

(c) For this $A$, use the $U$, $\Sigma$, and $V$ matrices to compute the pseudoinverse

$$
A^+ = (A^T A)^{-1} A^T = V \Sigma^{-1} U^T.
$$

(d) For this $A$, use (some or all) of $U$, $\Sigma$, and $V$ to compute the projector $AA^+$ onto $\mathcal{R}(A)$. 