1. [24 points: 8 points per matrix]
   Consider the three matrices
   
   \[
   \begin{bmatrix}
   2 & -1 \\
   -1 & 2
   \end{bmatrix}, \quad
   \begin{bmatrix}
   1 & 0 \\
   -1 & 3
   \end{bmatrix}, \quad
   \begin{bmatrix}
   -1 & 2 \\
   -1 & 1
   \end{bmatrix}.
   \]
   
   For each of these matrices, complete the following.
   (a) Compute by hand the matrix \((A - \lambda I)^{-1}\) for arbitrary \(\lambda\).
   (b) What are the eigenvalues of \(A\)?
   (c) What are the eigenvectors of \(A\)?

2. [22 points: (a) = 12 points; (b) = 5 points; (c) = 5 points]
   Arguably the most fundamental matrix in applied mathematics has the form
   
   \[
   A = \begin{bmatrix}
   0 & 1 & \quad & \quad & \quad & \quad \\
   1 & 0 & 1 & \quad & \quad & \quad \\
   \quad & 1 & 0 & \ddots & \quad & \quad \\
   \quad & \quad & \ddots & 1 & \quad & \quad \\
   \quad & \quad & \quad & \ddots & 1 & 0 \\
   \quad & \quad & \quad & \quad & 1 & 0
   \end{bmatrix},
   \]
   
i.e., the matrix has ones on the first super-diagonal and sub-diagonal, with zeros everywhere else.
   (a) Let \(A\) be an \(N \times N\) matrix of this form. Use trigonometry to verify that, for all \(j = 1, \ldots, N\),
   
   \[
   \lambda_j = 2 \cos \left( \frac{j\pi}{N+1} \right), \quad \mathbf{v}_j = \begin{bmatrix}
   \sin \left( \frac{j\pi}{N+1} \right) \\
   \sin \left( \frac{2j\pi}{N+1} \right) \\
   \vdots \\
   \sin \left( \frac{Nj\pi}{N+1} \right)
   \end{bmatrix},
   \]
   
   form an eigenvalue–eigenvector pair for \(A\). That is, show that \(A\mathbf{v}_j = \lambda_j \mathbf{v}_j\) for \(j = 1, \ldots, N\).
   (b) Later in the semester we will encounter a slight variation on this matrix,
   \[
   D = \begin{bmatrix}
   -2 & 1 & \quad & \quad & \quad & \quad \\
   1 & -2 & 1 & \quad & \quad & \quad \\
   \quad & 1 & -2 & \ddots & \quad & \quad \\
   \quad & \quad & \ddots & 1 & \quad & \quad \\
   \quad & \quad & \quad & \ddots & 1 & 0 \\
   \quad & \quad & \quad & \quad & 1 & -2
   \end{bmatrix},
   \]
   
   The matrix \(D\) has the same eigenvectors as \(A\). What are the eigenvalues of \(D\)?
   (c) For \(N = 32\), plot the eigenvectors of \(j = 1, 2, 31, 32\). (Plot the entry number \(k = 1, \ldots, 32\) along the horizontal axis, and the entries \((\mathbf{v}_j)_k\) of the eigenvector along the vertical axis. Produce one plot each for \(j = 1, 2, 31, 32\).)
Recall our earlier example of mechanical trusses at static equilibrium. Our four-step modeling process led to the equation \( A^*KAx = f \). Thus far we have considered situations where we knew how the structure was connected (encoded in \( A \)), the stiffness of the struts (the diagonal of \( K \)), and the loading force (\( f \)), and we sought the unknown displacements \( x \).

Many engineering applications present a different version of this problem: We can still see how the structure is connected, but we do not know the stiffness parameters for the struts. To find these parameters, we can apply different loads \( f \) to the structure, and measure the resulting displacements \( x \).

Are such experiments enough to determine the stiffness values \( k_1, \ldots, k_m \)?

Consider this approach. Start with \( A^*KAx = f \). Let \( b = Ax \), which we can compute if we are given \( A \) and then measure \( x \) for a given \( f \). This is the key observation:

\[
KAx = Kb = \begin{bmatrix} k_1 & \cdots & k_m \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} k_1b_1 \\ \vdots \\ km_b_m \end{bmatrix} = Bk,
\]

where \( B = \text{diag}(b) \) is the diagonal matrix whose \((j,j)\) entry equals \( b_j \), and \( k \) is the vector with \( k_j \) in its \( j \)th position. Then \( A^*KAx = f \) is equivalent to

\[
(A^*B)k = f,
\]

which we hope to solve for the unknown stiffness values \( k \).

(a) If the truss has \( n \) nodes (hence \( 2n \) degrees of freedom for planar motion) and \( m \) struts, what size is the matrix \( A^*B \)?

(b) Suppose that \( m > 2n \) (ensuring the truss is stable: we are unlikely to perform this study on an unstable - collapsed - truss). What do your answer to part (a) and the Fundamental Theorem of Linear Algebra tell you about the solvability of \((A^*B)k = f\) for the unknown \( k \)? Specifically, given the size of \( A^*B \), do you expect this equation to have a solution for all \( f \)? Explain. (Assume that \( x \) is polluted with noise, so it has no special properties that ensure \( A^*Bk = f \) has a solution, beyond what can potentially be gleaned from the dimensions of \( A^*B \).)

Suppose the bridge that appeared in Problem Set 2 supports train traffic crossing the New River. The struts are labeled with roman numbers, the nodes with bold-italic numbers, e.g., \( f_{12} \) applies horizontal force to the top-left node, \( \theta \).

You have been asked to analyze whether this old bridge need repair, by assessing the stiffness values \( k \) of these aging struts. You can experiment on the bridge by applying any desired load \( f \) to the structure, and measuring the corresponding vertical and horizontal displacements \( x \) of the nodes. Your instrumentation measures \( x \) to about a 1% relative error; this noise will compromise your results.

To simulate these measurements, you are given the bridge.p MATLAB code (on the class website). You collect data by creating a vector of loads \( f \) and calling \( x = \text{bridge}(f) \). (So you don’t peek at the code, we have compiled the MATLAB .m file as an executable but unreadable .p file.)

(c) Construct a vector \( f \), compute \( x \) using bridge.p, construct \( b = Ax \), and then solve the least squares problem

\[
\min_{k \in \mathbb{C}^m} \| (A^*B)k - f \|
\]

using MATLAB’s \( \backslash \) command \((k = (A^*B)\backslash f)\). Explain the physical interpretation of your choice of \( f \), and show the resulting estimated \( k \) values in a bar chart (MATLAB’s \texttt{bar} command).
(d) The noise in the measurements can cause significant errors in the estimated value of $k$. To obtain a better estimate, we can combine the results of several experiments. For example, if you conduct experiments with two different loads $f^{(1)}$ and $f^{(2)}$ giving measured displacements $x^{(1)}$ and $x^{(2)}$, then you hope

$$A^*B^{(1)}k = f^{(1)}, \quad A^*B^{(2)}k = f^{(2)},$$

where $B^{(1)} = \text{diag}(Ax^{(1)})$ and $B^{(2)} = \text{diag}(Ax^{(2)})$. You can find the $k$ that best satisfies these two equations simultaneously as

$$\min_{k \in \mathbb{C}^m} \left\| \begin{bmatrix} A^*B^{(1)} & A^*B^{(2)} \end{bmatrix}k - \begin{bmatrix} f^{(1)} \ f^{(2)} \end{bmatrix} \right\|_2.$$

Following this example, combine the measurements $x^{(j)}$ obtained from bridge.p for at least three different loads $f^{(j)}$ to determine an optimal $k$. Again describe the physical meaning of your choice of $f^{(j)}$, and show your estimated $k$ using a bar chart. Do your results look more reasonable than those obtained in part (c)?

(e) Suppose that you should consider the bridge unsafe if any of the struts have a stiffness value $k_j$ smaller than 0.25. Is our bridge faulty? If so, name the problematic strut(s). The bridge will be very expensive to repair, so if you find it unsafe, explain why you have confidence that your findings are accurate, not simply due to the error in your measurements.

This set-up describes the first “inverse problem” we have seen this semester: using experiments and measurements to infer the material properties of a system. It is adapted from a tissue biopsy example of Steve Cox.

4. [24 points: (a), (b) = 9 points each; (c), (d) = 3 points each]

Recall the sports ranking procedure described in lectures. The initial ranking $s^{(0)}_1, \ldots, s^{(0)}_n$ of $n$ teams is refined according to the formula

$$s^{(1)}_j = \frac{1}{n_j} \sum_{k=1}^n \alpha_{j,k} s^{(0)}_k,$$

where

$$\alpha_{j,k} = \begin{cases} 1, & \text{team } j \text{ defeated team } k; \\ 1/2, & \text{team } j \text{ tied team } k; \\ 0, & \text{team } j \text{ lost to team } k; \\ 0, & \text{team } j \text{ never played team } k; \end{cases}$$

and $n_j$ is the number of games team $j$ played. (Higher ranks should correspond to better teams.)

We encoded these equations in the form $s^{(1)} = As^{(0)}$, then improved the rankings with the recurrence

$$s^{(k)} = As^{(k-1)}$$

so that

$$s^{(k)} = A^k s^{(0)}.$$

We saw these rankings eventually reach an equilibrium where the ordering of the teams does not change as $k$ is further increased, and this optimal ranking could be found more directly by computing the eigenvector of $A$ corresponding to the eigenvalue having the largest magnitude (absolute value).

The model above is very simple – it assumes teams play each other at most once; it takes no account of margin of victory or whether the winning team played at home or away, for example.

For this problem, your open-ended task is to improve upon this model by making the $\alpha_{j,k}$ parameters more sophisticated. (You should do more than simply changing the number of points awarded for a win versus tie, for example.)
(a) Describe your variation of the model. How do you define the $\alpha_{j,k}$ parameters? What extra effect(s) do you take into account that were lacking in the model presented in class? Your model should allow teams to play each other multiple times (e.g., home and away).

(b) Implement your new model with the data available on the class website (the 2013 NFL season, 2012–2013 English Premier League season, or 2012–2013 NBA season). If you prefer, you can collect other data from your favorite sport, or bring in other extra data (e.g., time of possession, turnovers, etc.). Show how your rankings evolve over 15 iterations of the ranking process $(s^{(0)}, \ldots, s^{(14)})$. You may edit the code from `Sports_Rankings.m` from the class webpage and produce a similar plot.

(c) What are the eigenvalues of your $A$ matrix? (Use `eig` in MATLAB.)

(d) Compare the ranking you deduce from $s^{(14)} = A^{14}s^{(0)}$ with the rank suggested by the eigenvector of $A$ corresponding to the largest magnitude (absolute value) eigenvalue. Do they agree?