1. [30 points: (a) = 14 points; (b), (d) = 4 points; (c) = 8 points]

Consider the matrix
\[ A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \]

(a) Construct bases (by hand) for the four fundamental subspaces associated with \( A \):
\( \mathbb{R}(A) \), \( N(A^*) \), \( \mathbb{R}(A^*) \), and \( N(A) \).

(b) Characterize each of these four subspaces as a point, line, plane, etc.
Specify whether each space is contained in \( \mathbb{C}^2 \) or \( \mathbb{C}^3 \).

(c) Use the MATLAB scripts \texttt{plotline2}, \texttt{plotplane2}, \texttt{plotline3}, and \texttt{plotplane3} (available on the class web site) to create two illustrations: in the first, show both \( \mathbb{R}(A) \) and \( N(A^*) \); in the second, show both \( \mathbb{R}(A^*) \) and \( N(A) \).
You do not need to include your MATLAB commands for this problem, but please LABEL THE SUBSPACES ON YOUR PLOTS. It is easiest to print the plots and add the labels by hand. Use the \texttt{axis equal} command to scale all the axes with the same units.

(d) Indicate the orthogonality of the subspaces in your plots.

2. [21 points: 7 points per part for (a), (b), and (c)]

Write down a matrix with the required property or explain why no such matrix exists.

(a) The column space contains
\[ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \]
while the row space contains
\[ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \]

(b) The column space equals
\[ \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \]
while the null space equals
\[ \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}. \]

(c) The column space is \( \mathbb{C}^4 \), while the row space is \( \mathbb{C}^3 \).
[This problem is from Steve Cox’s \textit{Matrix Analysis in Situ}.]
3. [19 points: 13 points for (a); 6 points for (b)]
   Recall that \( \|v\|^2 = v^*v \) for any vector \( v \in \mathbb{C}^n \).
   In high school geometry, you might have learned this fact about parallelograms: the sum of the squares of the lengths of the four sides equals the sum of the squares of the lengths of the two diagonals.
   (a) Show that this holds in \( n \)-dimensional space by verifying the identity
   \[
   \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.
   \]
   (b) Explain (with a simple picture in two-dimensional space) how the identity you proved in part (a)
corresponds to the statement above about parallelograms.

4. [30 points: 6 points per part for (a)–(e), plus +4 bonus points for (f)]
   In class we will discuss that \( P = uu^* \) is a projector when \( \|u\|^2 = u^*u = 1 \).
   This problem will build up projectors onto higher dimensional subspaces from elementary projectors.
   (a) Suppose \( u_1^*u_1 = u_2^*u_2 = 1 \). You already know that \( P_1 = u_1u_1^* \) and \( P_2 = u_2u_2^* \) are projectors.
   Show that if \( u_1 \) and \( u_2 \) are orthogonal, then \( P = P_1 + P_2 \) will also be a projector.
   (b) Suppose \( P = P_1 + P_2 \) is a projector, as in part (a). What is \( \mathcal{R}(P) \)?
   (c) Use the above observations to construct a projector onto the two dimensional subspace
   \[
   S = \text{span} \left \{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right \}.
   \]
   (d) Show that if \( P \) is a projector, so too is \( I - P \).
   (e) For the projector in part (c), compute \( I - P \). Find \( u_3 \) such that \( I - P = u_3u_3^* \).
   (f) [Challenge problem: 4 bonus points]. Let \( P \in \mathbb{C}^{n \times n} \) be a projector.
   (i) Show that if \( P = P^* \), then \( \|Px\| \leq \|x\| \) for all \( x \in \mathbb{C}^n \).
   (ii) Show that if \( P \neq P^* \), then there exists some \( x \in \mathbb{C}^n \) for which \( \|Px\| > \|x\| \).